Unitary groups: basic definitions.

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1 Hermitian sesquilinear forms.

Let K/F be a separable quadratic extension of fields. Let Γ be $\mathrm{Gal}(K/F)$ and let c denote its non-trivial element. The general sesquilinear form on K^n is of the form $(x,y) = c(x)^t.J.y$ with x,y column vectors, and $J \in M_n(K)$. For the form to be hermitian we need $c(x,y) = (y,x)$, and this is equivalent to $J^t = c(J)$. Such a thing can be diagonalised if the characteristic of F isn't 2; let's stay away from characteristic 2 in what follows, because it currently doesn't interest me. It seems to me that diagonalising J gives elements of F which are well-defined up to elements of $N(K^\times)$, the norm subgroup of $F^\times$. So, for example, if F is the reals then J has a signature, which turns out to be well-defined—I think that the underlying fact that signatures work is because the sum of two norms is a norm, if $F = \mathbf{R}$. Watch out though! If F is a finite extension of $\mathbf{Q}_p$ with uniformiser ϖ and K is the unramified quadratic extension, then the forms defined by $\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ are isomorphic! The problem is that 1 and $\varpi - 1$ are both units in F, so they're norms from K, so if we choose $\beta \in K$ with norm $\varpi - 1$ and set $x = (1,\beta)^t$ and $y = (-c(\beta),1)^t$ then on the Hermitian form corresponding to the identity matrix we have $(x,x) = p = (y,y)$ and $(x,y) = 0$, and x and y give a basis of K^2.

2 Unitary groups.

Given $J \in \mathrm{GL}_n(K)$ with $J^t = c(J)$ (equivalently, given a non-degenerate Hermitian sesquilinear form) we define the unitary group $G := U(J)$ over F to be the group whose R-points, for R an F-algebra, are

$$\{g \in \mathrm{GL}_n(R \otimes_F K) : g^{ct}Jg = J\}$$

where g^{ct} denotes conjugate-transpose, with "conjugate" meaning fix R but let c act on K. Abstractly we're just asking for endormorphisms $g : K^n \to K^n$ such that $(gx, gy) = (x, y)$, so they preserve the Hermitian form.

Note that if R is a K-algebra then it gets an induced structure of an F-algebra and we can say a little more about $G(R)$. For $R \otimes_F K = R \otimes_K (K \otimes_F K) = R \oplus R$, this isomorphism induced by the natural map $K \otimes_F K = K \oplus K$ sending $k_1 \otimes k_2$ to $(k_1k_2, k_1c(k_2))$. Note that, via this isomorphism, conjugation on $R \otimes_F K$ just becomes the map $(r_1, r_2) \mapsto (r_2, r_1)$, and $G(R)$ is the pairs $(g_1, g_2) \in \mathrm{GL}_n(R) \times \mathrm{GL}_n(R)$ with $(g_2^t, g_1^t)(J, c(J))(g_1, g_2) = (J, c(J))$. Because $c(J) = J^t$ this just reduces to the one equation $g_2^tJg_1 = J$, so g_1 can be anything in $GL_n(R)$ and $g_2 = J^{-t}g_1^{-t}J^t$. In particular, the base change of G to K is canonically GL_n.

Note in particular that $G(K)$ is canonically $\mathrm{GL}_n(K)$ [because $K \otimes_F K$ is canonically $K \oplus K$, in the sense that we can distinguish the factors: there's a natural map $K \to K$, namely the identity]. On the other hand, if C is just an arbitrary extension of F which happens to be isomorphic to K, then $G(C)$ is non-canonically $\mathrm{GL}_n(C)$; you only get an isomorphism if you choose an isomorphism $C = K$, and if you then change your mind and choose the other one then the isomorphism might change. For example if J is the identity then the isomorphism changes by inverse-transpose.

Note that the determinant map $\mathrm{GL}_n(K) \to \mathrm{GL}_1(K)$ induces a map $G \to U(1)$, where $U(1)$ is the unique unitary group of rank 1 associated to K/F; this is a torus with character lattice isomorphic to $\mathbf{Z}$ with c acting non-trivially.

3 The quasi-split case.

Let $J \in \mathrm{GL}_n(K)$ be the antidiagonal matrix with 1s down the antidiagonal; note that we don't want alternating $+1$s and -1s; such a matrix won't in general be Hermitian! [The alternating $+1$s and -1s come later on, when defining the L-group]. The associated unitary group G/F is quasi-split. In fact the nice thing about this choice of J is that the upper-triangular matrices form an algebraic subgroup which in fact is a Borel in G! For $\gamma \in G(F) \subseteq \mathrm{GL}_n(K)$ upper-triangular to be in G we need $\gamma^{ct} J \gamma J^{-1} = 1$, so γ "rotated 180 degrees" is the inverse of γ^{ct}! Working it out explicitly gives some mess.

Within this mess are the diagonal matrices, a maximal torus. If $\mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathrm{GL}_n(K)$, then for this to be in the unitary group we need $\overline{d}_i d_{n-i} = 1$ for all i. If we set $\mathbf{S} = \mathrm{Res}_{K/F} \mathrm{GL}_1$ then the maximal torus is is a bunch of $\mathbf{S}$s if n is even and these plus a $U(1)$ if n is odd.

4 The Galois action on the based root datum.

We explicitly work out the "subtle" Galois action. We work over K of course, because this splits everything. Over K we know G becomes GL_n so let's choose the usual: an upper-triangular Borel, and a diagonal torus. If $(d_1, d_2, \ldots, d_n)$ is an element of this torus, then a general character sends this element to $\prod_i d_i^{m_i}$, with $(m_i) \in \mathbf{Z}^n$.

Now let's explicitly compute $\mu_G(c)$ on (m_i). Well, c is now acting on $K \times_F K$ on the *first* factor, and one checks that in the $K \oplus K$ model this sends (x, y) to $(c(y), c(x))$. Hence the non-linear complex conjugation map on $G(K)$ (the one whose fixed points are $G(F)$) sends $g \in G(K) = GL_n(K)$ to $c(J^{-t} g^{-t} J^t) = J^{-1} g^{-ct} J$. In particular $c(B)$ is a random Borel in $GL_n(K)$. We need to conjugate this back to B, and we do this of course by first conjugating with J, to get us back to g^{-ct}, and then conjugating with Φ, the anti-diagonal matrix with alternating $+1$ and -1 up the antidiagonal. This sends B to B and T to T. Explicitly, it sends $(d_1, d_2, \ldots, d_n)$ to $(c(d_n)^{-1}, c(d_{n-1})^{-1}, \ldots)$. Now apply our character to get an element of K, and finally apply c once more, to relinearise things. We see that $(m_1, m_2, \ldots, m_n)$ gets sent to $(-m_n, -m_{n-1}, \ldots, -m_1)$. Hence this is the Galois action on the based root datum.

5 The L-group in the quasi-split case.

Let's carefully follow Borel. Now I'm implicitly assuming that F is local or global (but recall I'm also assuming $2 \neq 0$). Let's put ourselves in the quasi-split case. Now over K our group G becomes canonically isomorphic to GL_n, so let's choose the obvious Borel and maximal torus. Our "pinning", or whatever it's called, could be matrices which are $I_n + E_{i,i+1}$, where I_n is the identity matrix and $E_{i,j}$ is the matrix which is zero except in the i,jth entry, where it's 1. The quadruple consisting of $(\mathrm{GL}_n, B, T, \{x_i\})$ over K has an automorphism defined thus: let w be the matrix which is zero off the antidiagonal, and has alternating $+1$s and -1s up the antidiagonal. It doesn't matter if one starts with a $+1$ or a -1 because the upshot will be the same; we're only going to be conjugating things by w, so the sign will make no difference. Now define the automorphism of $\mathrm{GL}_n(K)$ thus: send g to $wg^{-t}w^{-1}$. Note that conjugation by w turns a matrix 180 degrees and also changes signs in a "checkerboard" pattern. Of course conjugation by w will not change the trace, so the signs it changes are not on the leading diagonal, so in particular all our pinning elements end up with changed signs when you conjugate by w, and then they change signs again when you apply inverse-transpose, so this funny map sends the pinning to itself (but reverses the order). If $n \geq 1$ then this is an outer automorphism of the data, because it's changing determinants on GL_n.

Unsurprisingly, this is the automorphism we're going to use to define the L-group. To see this explicitly we need to compute μ_G, which we do in the usual way: it suffices to see what $\mu_G(c)$ is. I computed this already in the previous section but let's whizz through it again. To compute it, we first observe that c sends B to B and T to T, as they're defined over F. But in terms of elements, if $g \in G(K)$ then c acts non-K-linearly on K and hence induces a non-K-linear automorphism of G which we check to be the endomorphism sending $g \in \mathrm{GL}_n(K)$ to $Jg^{-ct}J$. Explicitly, on the torus, c sends $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ to $\mathrm{diag}(c(d_n)^{-1}, c(d_{n-1})^{-1}, \ldots)$. One sees easily, as we expected, that the F-points of the torus are fixed. Now given an algebraic map $T(K) \to K^\times$, if we pre-compose with the funny non-K-linear thing above, and then compose on the right with c as well, we get a K-linear map again, and this is the induced K-action of c on T: it sends $(d_1, d_2, \ldots, d_n)$ to $(d_n^{-1}, \ldots, d_1^{-1})$. Because it fixed the Borel too, this gives an action of c on Ψ, the based root datum. Hence we get an action on the based root datum of the dual group, which is $\mathrm{GL}_n(\mathbf{C})$. We now have to work out an algebraic automorphism of $\mathrm{GL}_n/\mathbf{C}$ which induces the above funny map on the torus and preserves the Borel and the pinning, and lo and behold it's $g \mapsto wg^{-t}w^{-1}$.

6 Some Weil group representations!

Can we think of any maps $W_{\mathbf{R}} \to {}^LG$, in the quasi-split case?