Trivial remarks about tori.

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1 Tori over C.

Let T be a torus over $\mathbf{C}$. Its cocharacter group is $X_*(T)$ and its character group is $X^*(T)$. These are both finite free $\mathbf{Z}$-modules and there is a natural perfect pairing between them.

The observation I always have to work out again and again is that there's a natural isomorphism $T(\mathbf{C}) = \mathrm{Hom}_{\mathbf{Z}}(X^*(T), \mathbf{C}^\times)$, identifying $t \in T(\mathbf{C})$ with the map sending $\phi : T \to \mathrm{GL}_1$ to $\phi(t) \in \mathrm{GL}_1(\mathbf{C})$.

Another way of saying this is $T(\mathbf{C}) = X_*(T) \otimes_{\mathbf{Z}} \mathbf{C}^\times$.

2 Tori over an arbitrary field.

If T is a torus over an arbitrary field then its character group is still a lattice, and we can form the dual torus $\widehat{T}$, which is traditionally a complex torus with $X_*(\widehat{T}) = X^*(T)$ and $X^*(\widehat{T}) = X_*(T)$. We deduce

$$\widehat{T}(\mathbf{C}) = \mathrm{Hom}(X^*(\widehat{T}), \mathbf{C}^\times) = \mathrm{Hom}(X_*(T), \mathbf{C}^\times) = X_*(\widehat{T}) \otimes \mathbf{C}^\times = X^*(T) \otimes \mathbf{C}^\times.$$

One checks easily that a group homomorphism $X_*(T) \to \mathbf{C}^\times$ is the same as a $\mathbf{C}$-algebra homomorphism $\mathbf{C}[X_*(T)] \to \mathbf{C}$. Hence if $\widehat{T}$ is regarded as an algebraic variety over the complexes, we have

$$\widehat{T} = \mathrm{Spec}(\mathbf{C}[X_*(T)]).$$

3 Split tori over non-arch local fields.

Let F be non-arch local and let T be a split torus over F. The fundamental fact here is that $X_*(T) = T(F)/T(\mathcal{O})$, where $\mathcal{O}$ is the integers of F, the map being the following: given $\phi \in X_*(T)$, ϕ is a map $\mathrm{GL}_1 \to T$, and one evaluates it at a uniformiser; the resulting element of $T(F)/T(\mathcal{O})$ is well-defined. As a consequence we have $X^*(\widehat{T}) = T(F)/T(\mathcal{O})$.

4 Hecke algebras.

Let G be locally compact and totally disconnected, and possibly some finiteness/countability conditions, which are always satisfied for F-points of reductive groups, and let K be a compact subgroup. Fix a Haar measure on G, normalised such that $\mu(K) = 1$. The Hecke algebra $H(G, K)$ is just the bi-K-invariant functions from G to $\mathbf{C}$ with compact support, and with multiplication given by convolution.

5 Hecke algebras of tori.

The crucial observation here is that if T is a torus over a non-arch local F, and if we normalise Haar measure on $G = T(F)$ so that K, the maximal compact subgroup of G, has measure 1, and if (for $t \in T(F)$) we let c_t be the characteristic function of tK, then (compute the convolution) we have $c_s c_t = c_{st}$. Hence the Hecke algebra $H(T(F), K)$ is just the group ring $\mathbf{C}[T(F)/K]$, and more generally the E-valued Hecke algebra is just $E[T(F)/K]$ for E any subfield of $\mathbf{C}$.

6 Hecke algebras of split tori.

Same notation as the last section. If furthermore T is split, then $K = T(\mathcal{O})$, so we get

$$H(T(F), T(\mathcal{O})) = \mathbf{C}[T(F)/T(\mathcal{O})] = \mathbf{C}[X_*(T)].$$

In particular $H(T(F), T(\mathcal{O}))$ is the ring of functions on the algebraic variety $\widehat{T}$.

7 The unramified local Langlands correspondence for split tori over non-arch fields.

And now we can prove the unramified local Langlands correspondence for split tori: if π is an unramified representation of $T(F)$ then it's a representation of $T(F)/T(\mathcal{O})$, and hence a group homomorphism $X_*(T) \to \mathbf{C}^\times$, and hence a ring homomorphism $\mathbf{C}[X_*(T)] \to \mathbf{C}$, which gives us a character of the Hecke algebra $H(T(F), T(\mathcal{O}))$. But it also gives us a group homomorphism $X^*(\widehat{T}) \to \mathbf{C}^\times$, and hence an element of $\widehat{T}(\mathbf{C})$. Indeed, what we have here is a bijection between unramified πs, elements of $\widehat{T}(\mathbf{C})$, and maximal ideals of $H(T(F), T(\mathcal{O}))$.

8 The local Langlands correspondence for tori over C.

I talk about this a lot in my notes in `local_langlands_abelian`. Here's how it works. If T is a torus over $\mathbf{C}$ and $L = X^*(T)$ then for any abelian topological group W (for example, $\mathbf{C}^\times$, or $\overline{R}^\times$) there's a canonical bijection between $\Pi := \mathrm{Hom}(\mathrm{Hom}(L, W), \mathbf{C}^\times)$ and $R := \mathrm{Hom}(W, \mathrm{Hom}(\hat{L}, \mathbf{C}^\times))$ (all homs are continuous group homs). So if $W = k^\times$ for k a topological field, one sees that $\mathrm{Hom}(T(k), \mathbf{C}^\times) = \mathrm{Hom}(k^\times, \hat{T}(\mathbf{C}))$. The obvious map is from Π to R: given $\pi \in \Pi$ and $w \in W$ and $\hat{\lambda} \in \hat{L}$ we need an element of $\mathbf{C}^\times$; the idea is that we apply π to the element $\lambda \mapsto w^{\hat{\lambda}(\lambda)}$ of $\mathrm{Hom}(L, W)$ and this works. For details see `local_langlands_abelian`.