The Adjoint Functor Theorem.

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1 Introduction.

"The existence of free groups is immediate from the Adjoint Functor Theorem." Whilst I've long believed this statement to be true, I have not (until now) ever bothered to understand a precise statement of one of the Adjoint Functor theorems, so I thought it would be a good exercise to do so. My reference is Mac Lane's "Categories for the working mathematician". There I discovered something I vaguely knew already—there are several adjoint functor theorems. One version appears to be called Freyd's Adjoint Functor Theorem, or the General Adjoint Functor Theorem (p117 of Mac Lane). Another is called the Special Adjoint Functor Theorem (p125), and a corollary of this is called "the classical form of the Special Adjoint Functor Theorem", or the SAFT (p126). I've been told that the SAFT was motivated by the construction of the Stone-Cech compactification of a topological space—indeed SAFT gives this compactification with little difficulty).

My initial thought was to understand the SAFT on the basis that it could be the "easiest" to understand. But (see later on) I realised soon after trying to understand the statement of the theorem that it would not apply to the case of constructing free groups! It will construct Stone-Cech compactifications, but wouldn't do what I wanted, so I gave up and decided to go for the General Adjoint Functor Theorem.

Having established which version of the Adjoint Functor Theorem I wanted to understand, the problem now becomes one of unravelling the definitions. So let's go.

The way Mac Lane sets things up is as follows. He starts with a model of set theory and the assumption that there is a universe U in his model. A set is *small* if it's an element of U. His definition of a category is a set of objects and a set of morphisms, but these sets don't have to be small. For example, **Set** is the category of all small sets, **Grp** is the category of all small groups, and so on. He will frequently assume however that his categories have small hom-sets, however, which of course means that if $a, b \in C$ then $\mathrm{Hom}(a, b)$ is a small set.

Now to the statement of the the General Adjoint Functor Theorem, as found on p117 of Mac Lane. The theorem, also called Freyd's Adjoint Functor Theorem was formulated and popularised by Freyd in 1964. Recall that a *left adjoint* to a functor $G : A \to X$ is a functor $F : X \to A$ such that there are natural bijections between $A(Fx, a)$ and $X(x, Ga)$.

Theorem 1.1 (General Adjoint Functor Theorem)**.** *If A is small-complete and has small hom-sets, then a functor $G : A \to X$ has a left adjoint iff it's continuous and satisfies the SSC (see below).*

The SSC, or "solution set criterion", is the following: for all $x \in X$ there's a small set I and a set of objects $a_i : i \in I$ of A and a set of arrows $f_i : x \to G(a_i)$ such that every arrow $x \to G(a)$ can be written as a composite $h = G(t)f_i$ for some i, where $t : a_i \to a$ is an arrow in A.

Rather than talking about the definitions of the category-theoretic words in this theorem (continuous, small complete), I will talk about the SSC for a while, as it's more "mathematical". Let's consider for example the forgetful functor from groups to sets. Does this satisfy the SSC? Well it certainly does: if x is a set, one can use the set with one element for I, define a_1 to be the free group on x and f_1 to be the canonical map; this works. But of course this presupposes the existence of free groups! In fact what I've just given you here is a special case of the proof that if the functor has an adjoint then it satisfies the SSC (with I the set with one element). The whole point of course is that way before one knows the existence of free groups, one can check the SSC—if x is a set and $f : x \to G(a)$ is a map to the underlying set of a group a, then the subgroup generated by x has size bounded by the cardinality of x (or can be countably infinite if x is finite)—and hence one can consider all groups of cardinality at most this, and then let I run through all maps from x to all of these groups!

If we consider the forgetful functor from compact Hausdorff topological spaces to topological spaces, then here's the argument that the SSC holds. It suffices to show that if X is a subset of a compact Hausdorff topological space Y then the closure of X has cardinality at most 2^{2^X}. WLOG Y is the closure of X. Here's a map L from Y to the power set of the power set of X: define $L(y)$ to be the collection of subsets T of X with the property that y is in the closure of T. I claim that this is an injection; if z is another element of Y then choose disjoint opens U containing y and V containing z in Y; then the closure of $U \cap X$ certainly misses V and hence $U \cap X$ isn't an element of $L(z)$. On the other hand, $U \cap X$ is dense in U and hence $U \cap X$ is an element of $L(y)$. Hence L is an injection and we've got a bound on the cardinality of the closure of X and now we easily get an SSC, just take all compact Hausdorff topological spaces of size at most this cardinality and so on. . . .

2 Notation in the theorem.

My task now is to explain what small-complete is, what continuous means (it means that the functor preserves all small limits, which I'll explain), and to do enough to convince myself that this theorem applies in the free group case. By the way, Mac Lane comments that the free group application is good because it constructs free groups "without entering into the usual (rather fussy) explicit construction of the elements of [the free group on X] as equivalence classes of words in letters of X". I find it hard to believe that this comment isn't tongue-in-cheek! It might avoid this, but it doesn't avoid constructions in comma categories in the proof of the theorem!

Anyway, let's get down to the definitions. A category is *small* if it comprises of a small set of objects and a small set of morphisms.

Now some words on limits. If J and C are categories, then C^J is the category of functors $J \to C$. Note that if C has small hom-sets and J is small then C^J has small hom-sets. There's an obvious diagonal functor $\Delta : C \to C^J$. A *limit* for $F : J \to C$ is a universal arrow from Δ to F. In other words, it's an object $r = \lim(F) \in C$ (think of a projective limit) and a natural transformation $\Delta(r) \to F$ which is universal amongst natural transformations $\Delta(c) \to F$. In other words, for any $c \in C$ and any "natural transformation $\Delta(c) \to F$," that is, arrows from c to all objects of J such that all the triangles commute—maybe I should say that this is called a "cone to the base F from the vertex c"—there's a map $c \to r$ making everything commute.

A *small diagram* in a category C is a morphism $F : J \to C$ with J a small category. A category C is *small complete* if all small diagrams in C have limits in C. For example, **Set**, **Grp**, **Ab**, **Rng**, R-**Mod**, **Comp Haus** are all small complete.

A functor $H : C \to D$ *preserves small limits* if for all $F : J \to C$ (J small) with a limiting cone $\nu : b \to F$, hitting everything with H gives us a limiting cone $h\nu : Hb \to HF$. This is a bit more than sending limiting objects to limiting objects—it sends limiting diagrams to limiting diagrams. A functor is called *continuous* if it preserves all small limits. Note that a random nice-looking functor might not be continuous: the functor from **Set** to **Ab** sending a set to the free abelian group on the set, isn't continuous. Note however that if C is a category with small hom-sets and c is an object of C then $\mathrm{Hom}_C(c, -)$ is continuous.

If C is complete and $H : C \to D$ preserves all small products and all equalizers of parallel pairs, then H is continuous—this is an exercise on p114 of Mac Lane. It can be thought of as a "morphism" version of a corollary on p109 of Mac Lane, which says that if C is a category which has equalizers of all pairs of arrows, and all small products, then it's small-complete.

Remark 2.0.1. When typing this up, I noticed that on p118 of Mac Lane there is a "representability theorem", which gives necessary and sufficient conditions for a functor to **Set** to be representable. I wondered whether this would be useful for constructing universal elliptic curves! Of course, it shouldn't be, because people like Schlessinger and Artin were proving deeper (non-formal) representability results in specific cases and must have been aware of this general nonsense. To apply the theorem on p118 to elliptic curves, even before one checks the analogue of SSC, one needs that the opposite category to the category of schemes is small-

complete. But I don't think that this is true—even though the opposite of the category of schemes is close to the category of rings, I don't think it's close enough. When constructing p-divisible groups, for example, one really doesn't take the direct limit; over an affine base one could take the projective limit of the resulting global sections, and this would construct a limit that worked in the category of rings, but I don't think that this limit is a limit in the category of schemes. Here's a concrete direct system in the category of schemes: consider the direct system defining μ_{p^∞} over $\mathbf{Z}$ say. I think I once convinced myself that this had no limit in the category of schemes—but unfortunately I can't reconstruct my proof! Maybe I should chck this later?[1]. On the other hand, 1
quotients don't exist in the category of schemes and surely this is enough.

3 Examples.

1) Can I deduce the existence of free groups? Let $G : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor. A left adjoint will be $F : \mathbf{Set} \to \mathbf{Grp}$ such that $\mathrm{Hom}_{\mathbf{Grp}}(FS, K) = \mathrm{Hom}_{\mathbf{Set}}(S, K)$ for any group K and set S. So indeed it's what we're after. I remarked above that **Grp** is small-complete but to check that the forgetful functor is continuous, it's worth finding out *why* this is the case. That **Set** is small complete is clear—the obvious definitions work. One checks that the same obvious definitions work for **Grp**. The fancy way of saying this is that the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ *creates limits*, which is stronger than saying it preserves them! The proof of this is just diagram-chasing.

Erm. . . that's it for examples because I've wasted too much time on this already!

4 Appendix: the uselessness of the Special Adjoint Functor Theorem.

I would have to define more terms if I wanted to give a full precise statement of the Special Adjoint Functor Theorem, so I will simply refer the reader to p125 of Mac Lane for the gory details. Here's the heart of the problem though. A small set Q of objects of a category C is called a *small cogenerating set* if for any $f \neq g : a \to b$ in C there is $q \in Q$ and a map $h : b \to q$ with $hf \neq hg$. For example, if $C = \mathbf{Set}$, T is a set with 2 elements, and $Q = \{T\}$, then Q is a small cogenerating set for C. The Special Adjoint Functor Theorem and its corollary apply to functors $G : C \to X$ with the property that C has a small cogenerating set. Unfortunately

Lemma 4.1. **Grp** *doesn't have a small cogenerating set.*

Proof. It suffices to construct abstract simple groups of arbitrarily large cardinality—as this would make any small generating set contain elements of arbitrarily large

[1]it would be nice to see an example.

cardinality in our universe, which can't happen. So the lemma follows from the assertion that $\mathrm{PGL}_2(k)$ is simple for any algebraically closed field k. This latter assertion is true—here's a proof. We have to show that the normal subgroup generated by any non-identity element is the whole thing. Take a non-identity element and let N be the normal subgroup it generates. One can conjugate the element so that it's upper-triangular, and some easy messing around now shows that N contains an element of the form $\binom{1,t}{0,1}$ with $t \neq 0$, and then that N contains all upper triangular unipotent matrices, and then all lower triangular unipotent matrices, and then all matrices in $\mathrm{SL}_2(k)$ with a 1 in the bottom right hand corner (each is an upper unipotent times a lower unipotent), and hence all matrices in $\mathrm{PGL}_2(k)$ with non-zero bottom right hand corner, and hence all of $\mathrm{PGL}_2(k)$ (use $\binom{1,1}{0,1}$). □

Hence we can't use the Special Adjoint Functor Theorem, or SAFT, to construct free groups. In fact this small cogenerating set condition is the one that's usually the hardest to check, and not always true.