Tannakian categories.

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1 Introduction

I just want to give definitions and examples.

2 Tensor categories.

If C is a category and $\otimes : C \times C \to C$ is a functor, then an *associativity constraint* for $(C, \otimes)$ is, for each $X, Y, Z \in C$, an isomorphism $X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ which is functorial in X, Y and Z and such that the two obviousish induced maps $X \otimes (Y \otimes (Z \otimes T)) = (X \otimes (Y \otimes Z)) \otimes T$ are the same: this is "the pentagonal axiom". A *commutativity constraint* for $(C, \otimes)$ is functorial (in X and Y) isomorphisms $X \otimes Y \to Y \otimes X$ such that the obvious induced endomorphism of $X \otimes Y$ (apply it twice) is the identity. The associativity and commutativity constraints are *compatible* if the two obviousish maps from $X \otimes (Y \otimes Z)$ to $(Z \otimes X) \otimes Y$ are the same (the "hexagon axiom"). An *identity obect* is a pair (U, u) with U an object of C and $u : U \to U \otimes U$ an isomorphism, such that $X \mapsto U \otimes X$ is an equivalence of categories.

A *tensor category* is $(C, \otimes)$ equipped with compatible associativity and commutativity constraints, and having an identity object (not prescribed).

Example: finitely-generated modules over a commutative ring. Warning: if you change the sign in the associativity constraint then the pentagon axiom fails!

Example: All modules over a commutative ring?

Example: The category of sets, with $\otimes$ being product of sets?

Basic properties: all identity objects in a tensor category are canonically isomorphic. If U is an identity object then there's a canonical isomorphism $X = U \otimes X$ for any X. All the natural ways of computing the tensor product of $n \geq 2$ objects are canonically isomorphic.

A *strictly full* subcategory of a category is a full subcategory (recall that full means that hom sets don't change) with the property that if X is in it, then anything isomorphic (in the big category) to X is also in it.

A *tensor subcategory* of a tensor category C is a strictly full subcategory which contains an identity (and hence all of the identities) and is closed under tensor products. Such a thing is, unsurprisingly, a tensor category itself.

3 Rigid tensor categories.

Say C is a tensor category, and let 1 be an identity object. If X, Y are objects of C, the functor $C^{opp} \to Set$ sending T to $\mathrm{Hom}(T \otimes X, Y)$ might perchance be representable, i.e., perhaps there's some Z such that $\mathrm{Hom}(T \otimes X, Y) = \mathrm{Hom}(T, Z)$. If it is, we call the representing object $\underline{\mathrm{Hom}}(X, Y)$. Note that the representing object comes with a canonical map $\underline{\mathrm{Hom}}(X, Y) \otimes X \to Y$, which is called "evaluation" and written $ev_{X,Y}$. Note also that $\mathrm{Hom}(1, \underline{\mathrm{Hom}}(X, Y)) = \mathrm{Hom}(X, Y)$.

Say $\underline{\mathrm{Hom}}(X,Y)$ exists for all $X,Y\in C$. Then we can define the dual $X^\vee$ of an object X by $X^\vee=\underline{\mathrm{Hom}}(X,1)$ and in fact $X\mapsto X^\vee$ is a contravariant functor $C\to C$. General nonsense shows us that there's a canonical map $X\to(X^\vee)^\vee$ and X is *reflexive* if this is an isomorphism.

Examples: In the category of abelian groups, $\mathbf{Z}/2\mathbf{Z}$ isn't reflexive because its dual is 0. In the category of vector spaces over a field, finite-dimensional vector spaces will be reflexive but infinite ones won't.

What we have so far isn't, apparently, enough to deduce that "$\underline{\mathrm{Hom}}$s commute with tensor products", that is, if $n\geq 2$ and X_i and Y_i are objects of C for $1\leq i\leq n$, then there's a natural map

$$\otimes_{1\leq i\leq n}\underline{\mathrm{Hom}}(X_i,Y_i)\to\underline{\mathrm{Hom}}(\otimes_{1\leq i\leq n}X_i,\otimes_{1\leq i\leq n}Y_i).$$

[As special cases we see that there are natural maps $(X_1^\vee\otimes X_2^\vee)\to(X_1\otimes X_2)^\vee$ and $X^\vee\otimes Y\to\underline{\mathrm{Hom}}(X,Y)$.]

A tensor category C is *rigid* if

(i) $\underline{\mathrm{Hom}}(X,Y)$ exists for all X and Y,

(ii) all objects are reflexive, and

(iii) all those maps above (commuting Homs and tensors) are isomorphisms. (it suffices to stick to the case $n=2$, unsurprisingly).

Nonexamples: sets, finite sets, finitely-generated modules over a general commutative ring, vector spaces over a field.

Examples: finite-dimensional vector spaces over a field. Finitely-generated *projective* modules over a commutative ring.

The axioms imply that the map $X\mapsto X^\vee$ is an equivalence of categories $C\to C^{opp}$ (and in particular that $\mathrm{Hom}(X,Y)=\mathrm{Hom}(Y^\vee,X^\vee)$, and that $uHom(X,Y)$ and $\underline{\mathrm{Hom}}(Y^\vee,X^\vee)$ are canonically isomorphic). In fact C^{opp} is naturally a tensor category and, when I've defined it, we'll see that this map is a tensor equivalence of tensor categories.

While we're here, let's define traces. For X an object of a rigid tensor category, there's a canonical map $\underline{\mathrm{Hom}}(X,X)\to X^\vee\otimes X\to 1$, where the last map is the evaluation map. Applying the functor $\mathrm{Hom}(1,*)$ to this map we get a map $\mathrm{Hom}(X,X)\to\mathrm{Hom}(1,1)$, that is, a map $\mathrm{End}(X)\to\mathrm{End}(1)$, and this is called the trace map. The *rank* of X is defined to be the trace of the identity map from X to X. The trace map $\mathrm{End}(1)\to\mathrm{End}(1)$ is the identity. The rank of $X\otimes Y$ is the rank of X composed with the rank of Y.

A *rigid tensor subcategory* of a rigid tensor category is a tensor subcategory (so strictly full, closed under tensor products) which is also closed under taking duals. One can check that such a thing is naturally a rigid tensor category.

4 Tensor functors.

A tensor functor $(C,\otimes)\to(C',\otimes')$ is a pair (F,c) with F a functor $C\to C'$ and c a collection of *isomorphisms* $c_{X,Y}:F(X)\otimes F(Y)\to F(X\otimes Y)$, functorial in X and Y, that "commute with the associativity and commutativity constraints" in the obvious way, and send identity objects to identity objects. If furthermore F is an equivalence of categories then F is said to be a *tensor equivalence.*

Example: the map $X\mapsto X^\vee$ from a rigid tensor category to itself is a tensor equivalence $C\to C^{opp}$.

One checks that if C and C' are rigid then the axioms force the induced map $F(\underline{\mathrm{Hom}}(X,Y))\to\underline{\mathrm{Hom}}(FX,FY)$ to be an isomorphism. Tensor functors commute with traces and ranks in the obvious way. In particular, in the example we'll come to at some point, where C is a certain category of finite-dimensional representations of a group, the forgetful functor (forgetting the group action) will be a tensor functor, so will preserve rank, and the rank of a finite-dimensional representation will just be its dimension.

A *morphism of tensor functors* $(F,c)\to(G,d)$ is a morphism of functors $\lambda:F\to G$ such that (i) $\lambda_1:F(1)\to G(1)$ is the isomorphism of identity objects between $F(1)$ and $G(1)$, and (ii) $\lambda_X\otimes\lambda_Y$ and $\lambda_{X\otimes Y}$ fit into the obvious commutative diagram involving c and d.

Amazing fact: if C and C' are rigid then any morphism of tensor functors $F, G : C \to C'$ is an isomorphism! The idea of the proof is that if $\lambda : F \to G$ then λ is, amongst other things, $\lambda_X : F(X) \to G(X)$ for all X, and we define $\mu : G \to F$ by letting μ_X be the dual of $\lambda_{X^\vee}$.

5 Abelian tensor categories.

An *additive tensor category* is a tensor category $(C, \otimes)$ such that C is additive and $\otimes$ is a bi-additive functor. An *abelian tensor category* is $(C, \otimes)$ such that C is abelian and $\otimes$ is bi-additive.

Example: finitely-generated projective modules over a general commutative ring R: this is additive and rigid, but not in general abelian. Finitely-generated modules over a general commutative Noetherian ring R: this is abelian, but not rigid in general. If $R = k$ is a field though then we get the finite-dimensional vector spaces over k and this is a rigid abelian tensor category.

If C is an additive tensor category then $\mathrm{End}(1)$ is a ring and, because there are canonical isomorphisms $X \to 1 \otimes X$ for each X, each X inherits an action of $\mathrm{End}(1)$. The action commutes with the endomorphisms of X, for all X, so "all homs are R-linear" and in particular R is commutative! Hence C is instantly R-linear and $\otimes$ is R-bilinear! If furthermore C is rigid then the trace morphism is an R-linear map $\mathrm{End}(X) \to R$.

Let C be a rigid tensor category that happens to be abelian. Then $\otimes$ is bi-additive, and commutes with direct and inverse limits, so in particular it's exact. This is because the map $X \mapsto X \otimes Y$ has a right adjoint (namely $Z \mapsto \underline{\mathrm{Hom}}(y, Z)$) so commutes with direct limits, and a duality argument gives the other way.

If C is rigid and abelian then there's a bijection between subobjects of 1 and idempotents in R. In particular if R is a field then 1 is simple. In fact an idempotent e in R gives a decomposition $C = C_1 \times C_2$ where C_1 is the objects of C on which e acts as the identity.

6 Rigid abelian tensor categories.

Recall that if C is a rigid tensor category which is also abelian, then $\otimes$ has behaves well with respect to additivity. We call such a category a *rigid abelian tensor category.*

Example: finite-dimensional vector spaces over a field. Here rank is the obvious thing.

We'll come to representation-theoretic examples in a minute, but let me give a stupid example first:

Example: Let k be a field, and let C be the category whose objects are pairs (V_0, V_1) of finite-dimensional vector spaces over k, or equivalently vector spaces graded by $\mathbf{Z}/2\mathbf{Z}$. We give C the structure of a tensor category whose associativity constraint is the obvious thing but whose commutativity constraint is given by the "Koszul rule of signs": we identify $v \otimes w$ in $V_i \otimes W_j$ with $(-1)^{ij} w \otimes v$ in $W_j \otimes V_i$. This is a rigid abelian tensor category, but the rank of (V_0, V_1) is $d_0 - d_1$ with $d_i = \dim(V_i)$, and in particular the rank is not always a non-negative integer! This means that this category can't be the category of representations of a group.

7 An important worked example.

If k is a field, and if G is *any* affine group scheme over k, in particular it's just the spectrum of a Hopf algebra A over k, with no assumptions on finite-generation of A as an algebra or anything, then the category of finite-dimensional (algebraic) k-representations of G is rigid and abelian.

Let's work some of this one out in detail. I know that "forgetting the G-action" is supposed to be a tensor functor to the category of k-vector spaces, and tensor functors really "commute with the tensors": $F(V \otimes W) = F(V) \otimes F(W)$. This means that the tensor structure on the category of finite-dimensional representations of G will have to be tensoring over k, not over $k[G]$ or anything like that, and we would like to define a G-action on $V \otimes_k W$ by defining $g(v \otimes w) := (gv \otimes gw)$. This doesn't really make sense because G is a group scheme, not a group, but here's how to make it work: if we think of the group as the Hopf algebra A, then a *comodule* for A is a vector space V over k equipped with k-linear $\rho : V \to V \otimes_k A$ such that the coidentity map $A \to k$ induces the identity map $V \to V \otimes_k A \to V \otimes_k k = V$ and such that the two obvious maps $V \to V \otimes_k A \otimes_k A$ (one using the comultiplication, one using ρ twice) are the same. One checks that to give a representation $G \to \mathrm{GL}(V)$ is just to give a comodule structure on V, and now one can do algebra, if one is so inclined, to see the real definitions of tensor product. I'm too lazy though, so will continually argue on points.

What next? There are obvious associativity and commutativity constraints. The trivial 1-dimensional representation is an identity object. The category is clearly abelian. Hom-sets present a subtlety! If V and W are representations of G then $\mathrm{Hom}(V, W)$ is the G-homs from V to W, which does *not* have a natural action of G, but for the internal homs we want $\mathrm{Hom}_G(T \otimes_k X, Y) = \mathrm{Hom}_G(T, \underline{\mathrm{Hom}}(X, Y))$ and because tensors are over k but homs are over G we see that we want $\underline{\mathrm{Hom}}(X, Y)$ to be $\mathrm{Hom}_k(X, Y)$ equipped with the usual action $(g.f)(x) = g(f(g^{-1}x))$; now if T is the trivial 1-dimensional representation we recover the fact that the G-homs from X to Y are the G-homs from the trivial representation to the k-homs!

This certainly now looks like a rigid abelian tensor category, doesn't it.

One nice fact about not-necessarily finitely-generated Hopf algebras A over k is that any finite subset of A is contained within a sub-Hopf-algebra that is finitely-generated as an algebra. This is rather delicate (but fun) to check! See 2.6 of Deligne-Milne. As a result, any Hopf algebra is a direct limit of Hopf algebras of finite type, so any affine group scheme over k is a projective limit of algebraic groups over k.

8 Neutral Tannakian categories.

If k is a field then a *neutral Tannakian category* over k is a rigid abelian tensor category C which is k-linear, and for which there exists an exact faithful k-linear tensor functor ω from C to the category of finite-dimensional vector spaces over k. Recall that *faithful* just means that the maps on the hom-sets are injective. Note that we don't specify ω in the data. Any such ω is called a *fibre functor* for the category.

Example: if k is a field, A is a Hopf algebra over k and C is the category Vec_k of finite-dimensional k-representations of the associated affine group scheme over k, then C is a neutral Tannakian category, and ω is just the forgetful functor. We call this category $\mathrm{Rep}_k(G)$.

Recall that any morphism of tensor functors between rigid tensor categories is an isomorphism, so the automorphisms of ω form a group, well, if they form a set! Let's try and figure out the automorphisms of ω in the situation above. For any k-rep X of G we need $\lambda_X : X \to X$ a k-linear map, such that λ_1 is the identity, $\lambda_{X \otimes Y} = \lambda_X \otimes \lambda_Y$, and also λ has to commute with G-equivariant maps $X \to Y$. Clearly any element of $G(k)$ will give such a λ. Now my understanding is that the converse is also true!

Even better: if R is a k-algebra then $V \mapsto V \otimes_k R$ is a tensor functor from finite-dimensional k-vector spaces to finitely-generated R-modules, and we can compose ω with this map and try and compute the automorphisms of the resulting functor: this automorphism group contains $G(R)$ and is in fact equal to $G(R)$. So we can recover the entire functor G from automorphisms of the fibre functor! In particular we can recover G from $\mathrm{Rep}_k(G)$ and ω.

Theorem 1. *Let C be a rigid abelian tensor category such that $k = \mathrm{End}(1)$ is a field, and let $\omega : C \to Vec_k$ be an exact faithful k-linear tensor functor. Then the automorphism group of ω (regarded as a functor from k-algebras to groups) is representable by an affine group scheme G, and ω can be re-interpreted as a map $C \to \mathrm{Rep}_k(G)$ (because G is acting on ω, as it were).*

Remarks: The proof looks like this. First construct the vector space A with a coalgebra structure on it (without using much at all). Then, using the tensor structure on C, define an algebra structure on A. Then, using rigidity of C show that it's a Hopf algebra (rather than a monoid scheme).

9 How do reductive groups fit into all this?

Let k be a field of characteristic zero and let G be an affine group scheme over k.

1) G is connected iff for any representation X of G on which G acts non-trivially, the strictly full subcategory of $\mathrm{Rep}_k(G)$ whose objects are isomorphic to subquotients of X^n, $n \geq 0$, is *not* stable under $\otimes$. For if you can find such an object (for which it's stable) then you get a surjection $G \to H$ with H finite and non-zero.

Again assume k has char 0. Let G^0 denote the identity component of G.

2) G^0 is a limit of reductive groups iff $\mathrm{Rep}_k(G)$ is semisimple! Here's the idea of the proof. Reduce to the case G of finite type. Reduce to k algebraically closed. Reduce to considering Lie algebras. Now it all follows from classical theorems.

10 More examples.

Finite-dimensional $\mathbf{Z}$-graded vector spaces are a neutral Tannakian category, and the obvious fibre functor gives us $G = \mathbf{G}_m$.

Finite-dimensional real vector spaces V plus a decomposition $V \otimes_{\mathbf{R}} \mathbf{C} = \oplus V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$ are a rigid tensor category and indeed a neutral Tannakian category; the obvious fibre functor gives us $\S := \mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m$.

More generally, say an affine algebraic group G over a field k is of *multiplicative type* if its character group is finitely-generated and if it's the dual of its character group M. So it's $0 \to T \to G \to C \to 0$ with C finite and of multiplicative type, and T a torus. A representation of G is just a vector space over k with an M-grading over $\overline{k}$ such that the grading is permuted in the obvious way by Galois.

TODO: (k-linear) Tensor functors. k-linear tensor cats. Rigid abelian tensor cats. And then I can define an ab cat and write down Deligne-Milne theorem 2.11.