Theorems and Counterexamples concerning slopes.

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April 26, 2012

Abstract

This highly informal document is just a collection of theorems about, and examples of points on, eigencurves for classical modular forms.

Written August 14th 2003.

Definitions: Fix $v : \overline{\mathbf{Q}}_p^\times \to \mathbf{Q}$, normalised such that $v(p) = 1$. We define $v(0) = \infty$. Write $M_k(\Gamma_1(N), \chi)$ for modular forms of weight k, level N and character χ, and $S_k(\Gamma_1(N), \chi)$ for cusp forms. Throughout, p is prime and $p \nmid N$. By the *slope* of an eigenform of level Np, we mean the p-adic valuation $v(a_p)$ of the eigenvalue a_p of U_p, where v is normalised such that $v(p) = 1$. We abuse notation and also talk about the *slope* of an eigenform of level N as being the p-adic valuation $v(a_p)$ of the eigenvalue of T_p, even though the "correct" thing to look at is the oldforms; this is just a notational convenience, because the "slope" of something at level N, if it is big, gives slightly more information than the slopes of the oldforms at level Np, and I have never worked out whether this extra information is worth anything but it's nice to have around.

If $\alpha \in \mathbf{Q}$ then let $d(N, p, k, \alpha)$ denote the number of eigenvalues of T_p on level N weight k trivial character cusp forms, with valuation α.

For a mod p semisimple representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, say that ρ is *locally reducible* if its restriction to a decomposition group at p is reducible, and *locally irreducible* otherwise. For a mod p or characteristic 0 eigenform f, say f is *locally reducible (resp. irreducible)* if ρ_f is, where ρ_f is the semi-simple mod p representation attached to f. Say that a component of the eigencurve is *locally reducible (resp. irreducible)* if the mod p representation (or pseudorepresentation) attached to one (equivalently, all) of its points is locally reducible (resp. irreducible). This isn't very good notation but this notion comes up again and again, so it's handy to use.

1 The Gouvêa-Mazur conjecture.

QUESTION 1: *Is it true that if $k_1, k_2 > 2\alpha + 2$ and $k_1 \equiv k_2$ mod $(p-1)p^n$ for some integer $n \geq \alpha$ then $d(N, p, k_1, \alpha) = d(N, p, k_2, \alpha)$?*

ANSWER 1: No. This was a conjecture of Gouvea and Mazur. Many counterexamples are known now: for example $N = 14, p = 5, \alpha = 1, k_1 = 6, k_2 = 26$; then $d(N, p, k_1, \alpha) = 1$ and $d(N, p, k_2, \alpha) = 2$.
QUESTION 2: *With notation as above, is it true that if $k_1 \equiv k_2$ mod $(p-1)p^n$ for some integer $n > \alpha$ (strictly greater), then $d(N, p, k_1, \alpha) = d(N, p, k_2, \alpha)$?*
ANSWER 2: I don't know and this is becoming computationally more difficult to check. My feeling is that one should work out conceptually what is going on (like Herrick appears to be doing) rather than just making minor changes to the conjecture and hoping for the best. For example the change indicated makes the resulting conjecture not strong enough to recover Hida theory in the ordinary case, so it's probably a bad choice.
QUESTION 3: *Is there some function $f(N, p, \alpha)$ such that with notation as above, if $k_1 \equiv k_2$ mod $(p-1)p^n$ for some integer $n \geq f(N, p, \alpha)$ then $d(N, p, k_1, \alpha) = d(N, p, k_2, \alpha)$?*
ANSWER 3: Yes; indeed there are explicit functions known. In [6] it is proved that one may take f to be a function which is quadratic in α. Buzzard (unpublished: see last corollary in §3 of [2]) proved an analogous result for automorphic forms on definite quaternion algebras over $\mathbf{Q}$ which were split at p. In both results, the function is of the form $A\alpha^2 + B\alpha + C$. In Buzzard's result, A is the size of the weight 2 cusp forms of level Np and I suspect that a similar thing will be true in Wan's case but haven't checked. Wan worked out the case $N = 1$ and $p \equiv 1$ mod 12 explicitly, and gets $A = (p+1)/6$. Note in particular that f depends on N and p as well as α, whereas the conjectured f in the Gouvêa-Mazur conjecture depended only on α. It is, as far as I know, still open as to whether one can construct an f that depends only on α.
QUESTION 4: *Restrict to the case $N = 1$. Is the Gouvêa-Mazur conjecture true then?*
ANSWER 4: No. Probably $N = 1$, $p = 59$, $k_1 = 16$, $k_2 = 3438$ and $\alpha = 1$ is a counterexample. In the unlikely event that it isn't, then for this N, p, k_1, k_2, we do know that some $\alpha < 1$ must give a counterexample instead. It's hard to compute at such high weights though. See [4] for more details.
QUESTION 5: *Is the Gouvêa-Mazur conjecture true when $N = 1$ and $p < 59$? For which p is the Gouvêa-Mazur conjecture expected to hold when $N = 1$?*
ANSWER 5: Computations indicate that the conjecture may hold for $N = 1$ and $p < 59$. I don't know what's going on for sure, but here are some ideas. Consider the following sets of primes: Let S_1 be the set of primes p for which the Gouvêa-Mazur conjecture is not true for prime p and level 1. Let S_2 be the set of primes p for which there is a classical level 1 eigenform with non-integral slope. Let S_3 be the set of primes p for which there is a weight k classical level 1 eigenform with slope $s > (k-1)/(p+1)$. Let T denote the set of primes p which are not $\mathrm{SL}_2(\mathbf{Z})$-regular, in the sense of [3]. That is, T is the set of primes p for which there is a non-ordinary level 1 cusp form of weight at most $p+1$ (equivalently, for which there is a locally irreducible component of the eigencurve: see notation above for the meaning of this). As far as I know, there is no prime p and no i in $\{1, 2, 3\}$ for which one can prove that $p \notin S_i$. There are

primes p for which one can prove $p \in S_i$ however, by exhibiting explicit forms. For example if $p = 59$ then $p \in S_1$ ([4]), $p \in S_2$ ([5]) and $p \in S_3$ ([5]). As far as I know, $p = 59$ is the smallest prime known to be in any of the S_i. Gouvêa in [5] raises the possibility that $S_2 = S_3$. Is it true that $S_1 = S_2 = S_3 = T$, or at least that $S_1 \subseteq S_2 = S_3 = T$? It is conjectured in [3] that $S_2 \subseteq T$, and the conjectures, plus some unrewarding combinatorics, will probably also imply that $S_1 \subseteq T$ and that $S_3 \subseteq T$ (I think I once checked that my conjectures implied that $2 \notin S_2$ and $2 \notin S_3$). Note that trivially $T \subseteq S_3$, and that, as far as I know, this is the only proven inclusion amongst any of S_1, S_2, S_3 and T. Finally note that T is computable, in the sense that given a prime p one can work out in finite time whether or not $p \in T$. In particular one knows that the primes in T which are at most 173 are $\{59, 79, 107, 131, 139, 151, 173\}$.
QUESTION 6: *For $N > 1$ and trivial character, what are the p for which the Gouvêa-Mazur conjecture holds?*
ANSWER 6: I don't know, but here are some comments. One can certainly define the analogue of S_1, S_2, S_3 and T above. One has to be careful with $p = 2$ in the definition of T in this generality, because if $N = 5$ then we definitely have $2 \in S_2$ (there is a level 5 weight 8 slope 3/2 eigenform) and $2 \in S_3$ (there is a level 5 weight 4 slope 2 eigenform, although note that the associated oldform of level 10 only has slope 3/2), but all components of the eigencurve are locally reducible. Note however that 2 is $\Gamma_0(5)$-irregular, with the definition in [3] so perhaps this is the correct definition to use. It's conjectured in [3] that if we define T to be the $\Gamma_0(N)$-irregular primes then $S_2 \subseteq T$, and perhaps the conjectures there and some combinatorics also imply that $S_1, S_3 \subseteq T$.
QUESTION 7: *If one fixes N and also a non-trivial character χ of level N, what are the p for which the natural generalisation of the Gouvêa-Mazur conjecture is expected to hold?*
ANSWER 7: Now I really don't know, and some natural generalisations of the things I mention above now really are false. Fix N and χ of level N, and let S_1, S_2, S_3 denote the obvious generalisations of S_1, S_2, S_3 above. Let T be the set of primes not dividing N for which the associated eigencurve only has components with locally reducible Galois representations. I have already indicated that this definition might not be a good one when $p = 2$. But here are some more problems. (I half-wonder whether these are occurring because the character is odd, and would be interested in counterexamples that occur for non-trivial even Dirichlet characters[1]). If $N = 4$ and χ is the Dirichlet character of conductor 4, and $p = 5$, then all components of the corresponding eigencurve have locally reducible Galois representations, so $5 \notin T$. But certainly $5 \in S_1$ because the number of slope 1 forms is 2 at weight 7 and 4 at weight 27. Also $5 \in S_2$ because there are slope 1/2 forms in weight 11. However perhaps $5 \notin S_3$? This does not contradict anything conjectured in [5] because here Gouvêa sticks to level 1. The first time one sees forms of slope $1, 2, 3, 4, 5$ is in weights $7, 13, 19, 25, 31$ respectively, which is exactly what Gouvêa allows. This example really confuses me.

[1] Does Herrick know any? He seems to know a lot of examples of various phenomena...

QUESTION 8: *What more can you tell me about the examples where the Gouvêa-Mazur conjecture is known to fail?*
ANSWER 8: In all the examples I have seen, with trivial character, we have $k_1 \leq p+1$ and $k_2 = k_1 + p(p-1)$ and $\alpha = 1$. Gouvêa and Mazur conjecture that $d(N,p,k_1,1) = d(N,p,k_2,1)$. Strangely enough, in all the examples I have computed (only about 6), I have found that $2d(N,p,k_1,1) = d(N,p,k_2,1)$, and the counterexamples were constructed by searching for N, p, k_1 such that $d(N,p,k_1,1) > 0$. This indicates that "Herrick's programme" of finding structure even in the locally irreducible case still has every hope of succeeding: there still appears to be structure, even when the Gouvêa-Mazur conjecture is failing.

2 Breuil's conjecture.

Recall now a conjecture of Breuil. Let f be an eigenform of level N, weight k and slope s. Breuil makes a purely local conjecture (Conjecture 1.5 of [1]) which has global consequences of the following form: if $k \leq 2p$ then knowledge of the slope of f alone is enough to determine whether f is locally reducible or locally irreducible. These local conjectures are known for $k \leq p$ (use Fontaine-Laffaille). They appear to be open for $k = p+1$, but the global consequences are known[2] for $k = p+1$. So let's restrict to the case $p+2 \leq k \leq 2p$. For $k = p+2$ Breuil conjectures that f of weight k and level N is locally irreducible iff its slope is strictly between 0 and 1. For $p+3 \leq k \leq 2p$, Breuil conjectures that f as above is locally reducible iff its slope is 0 or 1.

QUESTION 9: *Does Breuil's conjecture shed any light on the conjectures of Buzzard or Herrick, or on the Gouvêa-Mazur conjecture?*
ANSWER 9: Not much, because it restricts itself to $k \leq 2p$. However, it does tell you some things. For example, here is a pleasing consequence. Let F be an ordinary form of level N and weight $2 \leq k \leq p-1$. Then F is locally reducible. Let f be the mod p reduction of F and define $g = \theta f$. Then the weight k_g of g satisfies $p+3 \leq k_g \leq 2p$. Let G be an eigenform lifting g. Then G is locally reducible and non-ordinary so Breuil conjectures that it must have slope 1. Observations like this are present in Buzzard's conjectures; Calegari calls this fact the existence of a "ghost θ". In fact, the local and global observations seem to have been made at about the same time (and probably in the same place, namely Paris).
QUESTION 10: *What is the generalisation of Breuil's conjecture to $k > 2p$?*

[2] This fact opens up the possibility of using global methods to prove the local conjecture in the case $k = p+1$: for example, if one fixes $p > 2$, one can ask the following question: is the set of eigenvalues of T_p on $S_{p+1}(\Gamma_0(N))$ dense in the integers of $\overline{\mathbf{Q}}_p$? If one could prove this, then one could probably resolve Breuil's conjecture affirmatively in the case $k = p+1$: some local work is still required, but people like Breuil and Berger seem to be confident that the local results needed to finish the proof are within reach. Note also that Colmez seemed appalled when I suggested to him that Breuil's conjectures could be attacked using global methods, so if anyone decides to work on this then don't tell Colmez! You should have seen his face!

ANSWER 10: I think this is a very interesting question! I do not know the answer. Breuil's conjectures imply that for $2 \leq k \leq 2p$, if f is an eigenform of level N with slope s with $0 < s < 1$, then f is locally irreducible. I do know that this is not true for $k = 2p + 1$; if $p = 5$ then there is localy reducible weight 11 slope 1/2 form at level 3. Worse—there is a weight 11 locally irreducible form of slope 1/2 at level 8 (the character is the one with kernel $\{1, 3\}$), so it is definitely no longer the case that the slope determines whether the form is locally reducible or irreducible. On the other hand, still for $p = 5$, if you try $k = 12$ and stick to trivial character, and try all levels prime to 5 from 1 to 81, one finds that the only slopes occurring are $\{0, 1/2, 1, 2, 3, \infty\}$ and the corresponding forms are all locally reducible for slopes 0 and 1, and locally irreducible otherwise. So perhaps for even k the situation might be better? Or perhaps this is a red herring.

QUESTION 11: *What is this "ghost θ" thing? To what extent is it true that if F is a characteristic 0 form with mod p reduction f, and G is a form lifting θf, then the slope of G is* 1 *plus the slope of F?*

ANSWER 11: Breuil's conjecture implies this, as I said already, if F is ordinary of weight k and $2 \leq k \leq p - 1$. It's not true if $k = p$; for example for $p = 5$ there is a form of weight 11, slope 1/2 and level 4 which is congruent to θ of an ordinary form of weight 5. Is it true in weight $p + 1$ though? One could ask the following question at weight $2p + 2$: is it true that if $p > 2$ and F is an eigenform of weight $2p + 2$ and slope s then F is locally reducible iff $s \in \{0, 1\}$? Computations so far have not contradicted this. Note also that if F is the weight 16 slope 1 form of level 1 and $p = 59$ and f is its reduction, then θf has weight 66 and there is a slope 2 form in weight 66.

Does anyone know any other questions, theorems, or examples are of of a similar flavour, to add to this document? KB, 14/8/3.

References

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