Everything I know about mod p local Langlands for GL_2.

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1 Introduction.

March 2006: Just back from the amazing conference in Palo Alto. I emailed Paskunas to explain to him why there must be more mod p representations of $\mathrm{GL}_2(F)$, F local, than the ones he discovered, at least when $F = \mathbf{Q}_{p^f}$ an unramified extension of $\mathbf{Q}_p$ of degree $f > 1$, and in doing so managed to make sense of the argument myself. The reason is that Fred et al's conjectures have been numerically verified in lots of cases—and indeed appear now to have been proved in many cases, and what Paskunas has constructed doesn't explain enough of these conjectures (yet). Somehow I am now coming to the conclusion that there is a "moral" reason why Paskunas can't have found enough representations: Paskunas attaches a representation to a pair (χ, χ^s) where χ is a mod p representation of the torus in $\mathrm{GL}_2(k)$, k finite, and s is the obvious involution of the torus—on the other hand whatever have such objects got to do with the Galois side of things? It's hard to see—this makes me even wonder if his construction is in any way natural from the point of view of mod p Langlands (although it clearly was natural from the point of view of representations of Hecke algebras à la Vignéras).

Here's an overview of why Paskunas hasn't constructed all representations. Change notation: let F now be a totally real field where p is inert (for simplicity), and let ρ be a continuous totally odd irreducible global mod p Galois rep, coming from a Hilbert modular form. Lift to a classical Hilbert modular form of weight 2 (and full level p structure, if you like). We get a system of eigenvalues, all lying in the integers $\mathcal{O}$ of a finite extension of $\mathbf{Q}_p$. Now go up the p-tower taking etale cohomology of the modular curves of full level p^n structure and with $\mathcal{O}$-coefficients. Take Take the inductive limit. Take the sub-piece where the Hecke operators away from p are acting via the given system of eigenvalues. This is a representation of $\mathrm{GL}_2(F_p)$. Reduce it mod p. This is, I think, the mod p representation $\pi(\rho)$ attached to ρ. Note that the construction is global but Emerton has reason to believe that π should only depend on ρ locally at p.

If Fred et al predict that ρ is modular of a weight V then they predict that the $K := \mathrm{GL}_2(\mathcal{O})$-homs from V to π are non-zero. So they predict that the $G := \mathrm{GL}_2(F_p)$-homs from $\text{c-ind}_{KZ}^G V$ to π are non-zero (assuming we've set things up so that the centre acts the same on both sides). Choose $0 \neq \psi$ in this space. Now the Barthel-Livné T is a G-equivariant endomorphism of $\text{c-ind}_{KZ}^G V$, so ψT is another G-linear map $\text{c-ind}_{KZ}^G V \to \pi$.

Let $I(1)$ be the things in K which are congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ mod the max ideal. Note that $V^{I(1)}$ is 1-dimensional by some standard fact about mod p representations of $\mathrm{GL}_2(k)$ for k finite, or by an explicit calculation. Let v_0 be a basis for it.

Matt claims the following:
(a) If V is 1-dimensional then $\psi T(v_0) = T_p \psi(v_0)$
(b) and if not then $\psi T(v_0) = U_p \psi(v_0)$
where $U_p = \sum_{i=0}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)$ and $T_p = U_p + \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$.

Assuming this, Paskunas can't have constructed enough representations because things don't add up. The main point is that Paskunas' representations each have $I(1)$-invariants of K-socle

2-dimensional, corresponding to 2 of Fred's weights, and in the irreducible case for $f = 2$ it can essentially never happen that both weights occur in Fred's list.

Let me check that $T = U_p$ before explaining further because I've never done this and it's crucial to the argument.

2 $T = U_p$.

Let F now be a local field, with integers $\mathcal{O}$ and residue field k. Set $\Gamma = \mathrm{GL}_2(k)$ and $K = \mathrm{GL}_2(\mathcal{O})$. Let Z be the centre of $G := \mathrm{GL}_2(F)$. Choose a uniformiser ϖ of F. If (σ, V) is an irreducible mod p representation of Γ then extend first to K and then to KZ by letting $\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)$ act as 1. By definition,

$$\text{c-ind}_{KZ}^G V$$

is the functions $f : G \to V$ which satisfy $f(kg) = \sigma(k)(f(g))$ for all $k \in KZ$ (and in particular their support is a union of cosets KZg) and such that the support is in fact a finite union of such cosets. Define a G-action by $(gf)(g') = f(g'g)$.

Notation: $[g, v]$ is the unique function in $\text{c-ind}_{KZ}^G V$ supported on KZg^{-1} and sending g^{-1} to v. It's Breuil's twist, not mine. One checks that $g[h, v] = [gh, v]$ and $[gk, v] = [g, \sigma(k)v]$.

Frobenius reciprocity shows that $\mathrm{End}_G(\text{c-ind}_{KZ}^G V) = \mathrm{Hom}_K(V, \text{c-ind}_{KZ}^G(V))$ which is a space of functions from V to functions from G to V, so it can be regarded as a space of functions $\beta : G \times V \to V$. The fact that the functions from G to V are in $\text{c-ind}_{KZ}^G V$ means that

$$\beta(kg, v) = \sigma(k)(\beta(g, v))$$

and the fact that the map is K-equivariant means that

$$\beta(g, kv) = \beta(gk, v).$$

We can hence also think of these βs as functions $\phi : G \to \mathrm{End}_{\overline{\mathbf{F}}_p}(V)$ (with compact support mod Z) such that $\phi(k_1 g k_2) = \sigma(k_1) \circ \phi(g) \circ \sigma(k_2)$ as endomorphisms of V.

Now let π be any representation of G over $\overline{\mathbf{F}}_p$, on which $\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)$ acts trivially. Fix $\psi : \text{c-ind}_{KZ}^G V \to \pi$ a G-equivariant map. Let's fix $0 \neq v_0 \in V^{I(1)}$. Now ψ is determined by $\psi([1, v_0]) \in \pi$.

Now let's fix V as a tensor product of symm reps, so in particular the monoid $M_2(\mathcal{O})$ acts naturally on V rather than just K. Let's assume that our local field has parameters e and f. Then our representation is given by an element of $\{0, 1, 2, \ldots, p-1\}^f$ (tensor producting and Frobenius and so on) and if we think of things as homogeneous polynomials of degree r in two variables with $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)(f(x, y)) = f(ax + cy, bx + dy)$ then v_0 is the product of the $x_i^{r_i}$ for $1 \leq i \leq f$. Furthermore we can think of $\overline{\mathbf{F}}_p v_0$ as being $\left(\begin{smallmatrix}* & 0\\ * & 0\end{smallmatrix}\right)V$.

Now let's focus explicitly on the endomorphism T of $\text{c-ind}_{KZ}^G V$ given by the $\phi : G \to \mathrm{End}_{\overline{\mathbf{F}}_p}(V)$ as above, whose support is $KZ\alpha^{-1}KZ = KZ\alpha KZ = KZ\overline{\alpha}KZ$ with $\alpha = \left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)$ and $\overline{\alpha} = \left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)$, and defined by $\phi(\overline{\alpha})(v) = \sigma(\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right))v$. Because we have a monoid action, we may equally well define ϕ uniquely by saying its support is $KZ\alpha KZ$ and $\phi(\alpha)v = \sigma(\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right))v$.

Let T denote the induced endomorphism of $\text{c-ind}_{KZ}^G V$. Now what is $\psi(T([1, v_0]))$?

In general, $T([1, v_0])$ is the function $G \to V$ sending g to $\phi(g)(v_0)$, and this function has support in $KZ\alpha^{-1}KZ$. For λ running through a set of representatives in $\mathcal{O}$ for $\mathcal{O}/\varpi$, define $g_\lambda := \left(\begin{smallmatrix}\varpi & \lambda\\ 0 & 1\end{smallmatrix}\right)$. These are Breuil's $g_{1,\lambda}^0$. We have

$$\begin{aligned} KZ\alpha^{-1}KZ &= KZ\alpha KZ \\ &= \Big(\coprod_{\lambda \in \mathcal{O}/\varpi} KZg_\lambda^{-1}\Big) \coprod KZ\alpha^{-1} \end{aligned}$$

and $\left(\begin{smallmatrix}0 & 1\\ 1 & -\lambda\end{smallmatrix}\right) g_\lambda \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right) = \alpha$ so $\alpha^{-1} = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right) g_\lambda^{-1} \left(\begin{smallmatrix}0 & 1\\ 1 & -\lambda\end{smallmatrix}\right)^{-1}$, and $\left(\begin{smallmatrix}1 & -\lambda\\ 0 & 1\end{smallmatrix}\right) g_\lambda = \overline{\alpha}$ so $g_\lambda = \left(\begin{smallmatrix}1 & \lambda\\ 0 & 1\end{smallmatrix}\right)\overline{\alpha}$.

Now because the monoid acts we can argue as follows:

$\phi(g_\lambda^{-1})(v_0) = \phi(\left(\begin{smallmatrix} 1 & -\lambda \\ 0 & \varpi \end{smallmatrix}\right))v_0 = \sigma(\left(\begin{smallmatrix} 1 & -\lambda \\ 0 & 0 \end{smallmatrix}\right))v_0 = v_0$, and $\phi(\alpha^{-1})v_0 = \phi(\overline{\alpha})v_0 = \sigma(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right))v_0 = 0$ *unless all the* r*s are zero.*

Conclusion: $T([1,v_0])$ is the function supported on $\coprod_\lambda KZg_\lambda^{-1}$ and sending g_λ^{-1} to v_0 (assuming not all the rs are zero). So $T([1,v_0]) = \sum_\lambda [g_\lambda, v_0] = \sum_\lambda g_\lambda[1,v_0]$.

This means that $\psi(T([1,v_0])) = \sum_\lambda g_\lambda \psi([1,v_0]))$. So indeed Matt is right: $\psi(Tv) = U_p\psi(v)$. Very nice!

3 Weights

Let's write down Fred at al's conjectures explicitly in the semisimple case, for small values of f. I think I'll skip $f = 1$. Notation: Let k be a field with p^f elements; fix an embedding into $\overline{\mathbf{F}}_p$ and call it τ_0. Now let τ_1 be $\mathrm{Frob} \circ \tau_0$ and so on, where Frob is $x \mapsto x^p$ on $\overline{\mathbf{F}}_p$. We attempt to incorporate all the standard notations, namely "$r+1 = b$ and $b+1 = k$". A weight will either be written $V_{\underline{a},\underline{b}}$, or simply as $(\underline{a},\underline{b})$, and this corresponds to the representation $\det^a \otimes \mathrm{Symm}^r = \det^a \otimes \mathrm{Symm}^{b-1}$, with vectors $\underline{a} = (a_0, a_1, \ldots, a_{f-1})$ and so on indexed by embeddings.

3.1 The reducible semisimple case.

If $\rho|D_p$ is reducible and semisimple then on inertia it looks like $\chi_1 \oplus \chi_2$, and we associate $V_{(\underline{a},\underline{b})}$ to ρ iff there is some subset J of S such that $\chi_1 = \prod_\tau \omega_\tau^{a_\tau} \prod_{\tau \in J} \omega_\tau^{b_\tau}$ and $\chi_2 = \prod_\tau \omega_\tau^{a_\tau} \prod_{\tau \notin J} \omega_\tau^{b_\tau}$.

We now work out the dictionary the other way around: given an explicit ρ, which Vs do we get? Say $\rho|D_p$ is reducible and $\rho|I_p = \chi \oplus 1$. Say $\chi = \prod_i \omega_i^{(c_i+1)} = \prod_i \omega_i^{d_i}$ with $d_i = 1 + c_i$. For any $J \subseteq \{0, 1, \ldots, f-1\}$ we need to solve

$$\sum_i a_i p^i + \sum_{j \in J} b_j p^j = \sum_i d_i p^i$$

$$\sum_i a_i p^i + \sum_{j \notin J} b_j p^j = 0.$$

Subtracting, we deduce

$$\sum_{j \in S} \chi_J(j) b_j p^j = \sum_i d_i p^i$$

where $\chi_J(j) = 1$ for $j \in J$ and -1 for $j \notin J$. We normalise things so that $1 \le b_j \le p$ and $0 \le a_j \le p-1$. Recall that $r = b-1$. Also $c = d-1$.

Algorithm:

1) Compute the b_i via $\sum_{j \in S} \chi_J(j) b_j p^j = \sum d_i p^i$

2) Compute the a_i via $\sum_i a_i p^i + \sum_{j \in J} b_j p^j = \sum d_i p^i$

3) Compute the r_i by subtracting 1 from the b_i and replacing $d_i - 1$ by c_i.

3.1.1 $f = 2$ reducible.

Solution:

J	b_0	b_1	a_0	a_1	r_0	r_1
$\{0,1\}$	d_0	d_1	0	0	c_0	c_1
$\{1\}$	$p - d_0$	$d_1 + 1$	c_0	$p-1$	$p-2-c_0$	c_1+1
$\{0\}$	d_0+1	$p - d_1$	$p-1$	c_1	c_0+1	$p-2-c_1$
$\{\}$	$p-1-d_0$	$p-1-d_1$	c_0+1	c_1+1	$p-3-c_0$	$p-3-c_1$

3.1.2 $f = 3$ reducible.

Solution:

J	b_0	b_1	b_2	a_0	a_1	a_2	r_0	r_1	r_2
$\{0,1,2\}$	d_0	d_1	d_2	0	0	0	c_0	c_1	c_2
$\{1,2\}$	$p-d_0$	d_1+1	d_2	c_0	$p-1$	$p-1$	$p-2-c_0$	c_1+1	c_2
$\{0,2\}$	d_0	$p-d_1$	d_2+1	$p-1$	c_1	$p-1$	c_0	$p-2-c_1$	c_2+1
$\{0,1\}$	d_0+1	d_1	$p-d_2$	$p-1$	$p-1$	c_2	c_0+1	c_1	$p-2-c_2$
$\{0\}$	d_0+1	$p-d_1$	$p-1-d_2$	$p-1$	c_1	c_2+1	c_0+1	$p-2-c_1$	$p-3-c_2$
$\{1\}$	$p-1-d_0$	d_1+1	$p-d_2$	c_0+1	$p-1$	c_2	$p-3-c_0$	c_1+1	$p-2-c_2$
$\{2\}$	$p-d_0$	$p-1-d_1$	d_2+1	c_0	c_1+1	$p-1$	$p-2-c_0$	$p-3-c_1$	c_2+1
$\{\}$	$p-1-d_0$	$p-1-d_1$	$p-1-d_2$	c_0+1	c_1+1	c_2+1	$p-3-c_0$	$p-3-c_1$	$p-3-c_2$

Remark: the reducible case is really the easy case as far as the combinatorics go. You can guess the general answer somehow. The irreducible case is messier because there is somehow some implicit asymmetry.

3.2 The irreducible case.

If $\rho|D_p$ is irreducible then for each τ there are two extensions of τ to the quadratic extension k' of k. Let S' denote the set of all embeddings $k' \to \overline{\mathbf{F}}_p$. There's a natural 2-to-1 map $\pi : S' \to S$.

The dictionary is that we will associate the representation $V_{(\underline{a},\underline{b})}$ to ρ iff there exists some subset J' of S' (Fred calls it J) which maps bijectively onto S via π and such that $\rho|I_p = \prod_\tau \omega_\tau^{a_\tau} \left(\begin{smallmatrix}\psi & 0\\ 0 & \psi^q\end{smallmatrix}\right)$ with $\psi = \prod_{\tau'\in J'} \omega_{\tau'}^{b_{\tau'}}$.

Again we wish to do the computation the other way around: given a ρ and a J' we need to unravel the as and bs. Let τ_0' be a lifting of τ_0 and let ρ be $\psi \oplus \psi^q$ with $\psi = \tau_0'^{d_0+pd_1+\ldots+p^{f-1}d_{f-1}}$. We have $J' \subset \{0,1,2,\ldots,2f-1\}$ of size f; for $0 \le j < f$ let $e_j = 1$ if $j \in J'$ and p^f if not. Then

$$(p^f+1)\sum_i a_i p^i + \sum_{0\le j<f} e_j b_j p^j = \sum_i d_i p^i.$$

Reducing mod p^f+1 gives

$$\sum_{0\le j<f} \chi_{J'}(j) b_j = \sum_i d_i p^i \bmod p^f+1$$

where $\chi_{J'}(j) = 1$ for $j \in J'$ and -1 for $j \notin J'$. Reducing mod p^f-1 gives

$$\sum_i (2a_i+b_i)p^i = \sum_i d_i p^i \bmod p^f-1.$$

3.2.1 $f = 2$ irreducible

Solution:

$J'\cap\{0,1\}$	b_0	b_1	a_0	a_1	r_0	r_1
$\{0,1\}$	d_0	d_1	0	0	c_0	c_1
$\{1\}$	$p-d_0$	d_1+1	c_0	$p-1$	$p-2-c_0$	c_1+1
$\{0\}$	d_0-1	$p-d_1$	0	c_1+1	c_0-1	$p-2-c_1$
$\{\}$	$p+1-d_0$	$p-1-d_1$	c_0	c_1+1	$p-1-c_0$	$p-3-c_1$

3.2.2 $f = 3$ irreducible

Solution:

$J' \cap \{0,1,2\}$	b_0	b_1	b_2	a_0	a_1	a_2	r_0	r_1	r_2
$\{0,1,2\}$	d_0	d_1	d_2	0	0	0	c_0	c_1	c_2
$\{1,2\}$	$p-d_0$	d_1+1	d_2	c_0	$p-1$	$p-1$	$p-2-c_0$	c_1+1	c_2
$\{0,2\}$	d_0	$p-d_1$	d_2+1	$p-1$	c_1	$p-1$	c_0	$p-2-c_1$	c_2+1
$\{0,1\}$	d_0-1	d_1	$p-d_2$	0	0	c_2+1	c_0-1	c_1	$p-2-c_2$
$\{0\}$	d_0-1	$p-d_1$	$p-1-d_2$	0	c_1+1	c_2+1	c_0-1	$p-2-c_1$	$p-3-c_2$
$\{1\}$	$p+1-d_0$	d_1+1	$p-d_2$	c_0	$p-1$	c_2	$p-1-c_0$	c_1+1	$p-2-c_2$
$\{2\}$	$p-d_0$	$p-1-d_1$	d_2+1	c_0	c_1+1	$p-1$	$p-2-c_0$	$p-3-c_1$	c_2+1
$\{\}$	$p+1-d_0$	$p-1-d_1$	$p-1-d_2$	c_0	c_1+1	c_2+1	$p-1-c_0$	$p-3-c_1$	$p-3-c_2$

4 Bardoe and Sin.

Let k be a finite field of cardinality $q = p^f$, let $\Gamma = \mathrm{GL}_2(k)$ containing the diagonal torus H, the upper triangular unipotent U, and the Borel $B = HU$. Bardoe and Sin (Jour LMS 61 (2000) 58–80) set $V = k^2$ with its natural left action of Γ; then the $\overline{\mathbf{F}}_p$-vector space $\overline{\mathbf{F}}_p[V]$ with basis V is a $k[\Gamma]$-module. For $0 < c < q-1$ (Note: Bardoe and Sin write d for c but this doesn't fit well with our notational conventions) let $A[c]$ denote the subspace of $\overline{\mathbf{F}}_p[V]$ consisting of functions $f : V \to \overline{\mathbf{F}}_p$ such that $f(\lambda \underline{v}) = \lambda^c f(\underline{v})$. Examples of elements of $A[c]$ are homogeneous polynomials of degree c, or degree $c + q - 1$ and so on.

Fix an embedding $k \to \overline{\mathbf{F}}_p$ and let χ be the character of H sending $\mathrm{diag}(\lambda, \mu)$ to μ^c. Extend to a character of B The easy lemma is that $\mathrm{ind}_B^\Gamma(\chi) = A[c]$. The proof is as follows: if B_1 denotes the subgroup $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ of B (note that this is contained in the kernel of χ) then the compact induction is, by definition, functions from $B_1 \backslash G$ to $\overline{\mathbf{F}}_p$ such that $f(hg) = \chi(h)f(g)$, with action $gf(g') = g(g'g)$. Now B_1 is the stabiliser of the row vector $(0,1)$ via the natural right action of Γ on row vectors, so $B_1 \backslash G$ is naturally $V \backslash \{(0,0)\}$; the left action of $\mathrm{diag}(\lambda, \mu) \in H$ on $B_1 \backslash G$ is multiplication by μ, and the result now follows from the definitions.

Now Bardoe and Sin tell us the entire submodule structure of $A[c]$. Let's assume that in the base p expansion $c_0 + pc_1 + \cdots + p^{f-1}c_{f-1}$ of c, none of the digits c_i are 0 or $p-1$, so we're well away from the walls. Then $A[c]$ has 2^f Jordan-Hoelder factors and here's an explicit formula for them: for any $(s_0, s_1, \ldots, s_{f-1}) \in \{0,1\}^f$ [note: Bardoe and Sin call it $(r_0, r_1, \ldots, r_{f-1})$ set $\lambda_j = c_j + ps_{j+1} - s_j \in [0, 2p-2]$ and let $L(s_0, \ldots, s_{f-1})$ be $S^{\underline{\lambda}}$, which has a rather subtle definition: it's the tensor product for $j = 0$ to $f-1$ of S^{λ_j} twisted by Frob^j, where S^{λ_j} is functions in $\overline{\mathbf{F}}_p[X_0, X_1]$ of total degree λ_j except that it's actually the image of these functions in $\overline{\mathbf{F}}_p[X_0, X_1]/(X_0^p, X_1^p)$ (note that the ideal is Γ-invariant so that these things really are representations of Γ. These are the J-H factors of $A[c]$.

Now we need to be able to read off what these representations S^λ really are. First the easy case If $\lambda \leq p-1$ then there is no mystery: S^λ is just the usual Symm^λ, of dimension $\lambda + 1 \in [1, p]$, and, because we always think of these things as homogeneous polys of degree λ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) X_0 = aX_0 + cX_1$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) X_1 = bX_0 + dX_1$ we deduce that the U-invariant functions, that is those such that $f(X_0, X_1) = f(X_0, X_0 + X_1)$, are 1-dimensional and spanned by X_0^λ.

If however $p - 1 \leq \lambda \leq 2p-2$ then S^λ has dimension $\lambda + 1 - 2(\lambda - (p-1)) = 2p - 1 - \lambda \in [1, p]$, and it's irreducible (according to Bardoe and Sin) but we have to think a bit to see what it is. In fact here's what it is. Say $0 \leq \lambda \leq 2p-2$. Multiplication defines a perfect Γ-invariant pairing $S^\lambda \times S^{2p-2-\lambda} \to S^{2p-2}$ which is 1-dimensional and on which Γ acts by $\det^{p-1}$ [proof: check it on a computer for $p = 7$ so it must be true] [better proof: the coefficient of x^{p-1} in $(ax+c)^{p-1}(bx+d)^{p-1}$ clearly has degree $2(p-2)$ and it's a 1-dimensional representation of the algebraic group GL_2 so it had better be $\det^{p-1}$]. Hence S^λ is the twist of the dual of $S^{2p-2-\lambda}$ by $\det^{p-1}$. Now if $\lambda \leq p-1$ then the dual of Symm^λ is obtained by letting g act as g^{-t}, the inverse transpose (this works because $\lambda < p$; it's a dirty hack) and because $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is conjugate to $\left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$ we deduce that the dual of Symm^λ is $\det^{-\lambda} \mathrm{Symm}^\lambda$. Hence for $\lambda \leq p-1$ we conclude that $S^{2p-2-\lambda} \cong \det^{p-1-\lambda} \mathrm{Symm}^\lambda$, or, equivalently, for $\lambda \geq p-1$, we conclude that $S^\lambda \cong \det^{\lambda-(p-1)} \mathrm{Symm}^{2p-2-\lambda}$. Note that if $\lambda = p-1$ then we conclude something that is true, which is a relief.

Note that the U-invariant vectors of S^λ can be spotted for $\lambda \geq p-1$ because $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) f(X,Y) = f(aX+cY, bX+dY)$ so we're looking for polynomials in $\overline{\mathbf{F}}_p[X_0, X_1]/(X_0^p, X_1^p)$ of degree λ such that $f(X,Y) = f(X, X+Y)$, and $X^{p-1}Y^{\lambda-(p-1)}$ works; note that this is quite different in general to the function X^λ, which is what works when $\lambda < p$.

We've now written down the J-H factors of $A[c]$ but we can do much better: we can write down the entire submodule lattice and identify the J-H factors of each submodule. If S is the partially ordered set $\{0,1\}^f$, ordered by componentwise $\leq$, then the map sending a submodule of $A[c]$ to the subset of S consisting of its J-H factors is a bijection between the submodules of $A[c]$ and the ideals of S (recall that an ideal of a poset is a subset S such that if $s \in S$ and $t \leq s$ then $t \in S$).

We see from this that $A[c]$ has a unique irreducible sub and a unique irreducible quotient. The sub corresponds to the homogeneous functions that are "twists of polynomials", i.e., functions $f : V \to \overline{\mathbf{F}}_p$ spanned by those of the form $f(x,y) = \prod_{0 \leq i < f} F_i(x,y)^{p^i}$ where F_i is homogeneous of degree c_i. The quotient is most easily seen by looking at things from the induced picture: we have $\mathrm{ind}_B^\Gamma(\chi)$ is functions $f : \Gamma \to \overline{\mathbf{F}}_p$ such that $f(bg) = \chi(b)f(g)$ and if ρ is the irreducible representation of Γ such rhat ρ^U is χ then one defines the map by sending the unique function supported on B and such that $f(1) = 1$ to a non-zero element of ρ^U.

Note that the sub has the property that its U-invariants are the functions f such that $f(x,y) = f(x, x+y)$ and so all the F_i are monomials in x and the U-invariants afford the representation of H defined by $\mathrm{diag}(\lambda,\mu) \mapsto \lambda^c$, so it's χ^s.

If one quotients out $A[c]$ by its socle, then the socle of the quotient will be the direct sum of f irreducible representations, corresponding to vectors $\underline{s} \in \{0,1\}^f$ with one non-zero entry. If $f = 1$ we know what's going on; $A[c]$ mod the socle is irreducible. For $f > 1$ the corresponding f representations will have parameters equal to $(c_1, c_2, \ldots, c_{i-1}+p, c_i - 1, c_{i+1}, \ldots, c_f)$ and changing c_j to $c_j + p$ changes S^{c_j} to S^{c_j+p} which is a twist of $S^{2p-2-(c_j+p)} = S^{p-2-c_j}$. So the effect is to change one c_j to $p-2-c_j$ and change the next one by adding 1. Important note to me: later on changing an adjacent c_i will subtract, not add, one to the λs of the form $p-2-c_j$.

Worked examples of this are easy—much easier than Fred's conjectures in the irreducible case! For each element $s \in S$ we work out the corresponding J-H factor using the usual notation $\det^{\underline{a}} \mathrm{Symm}^{\underline{r}}$. We decompose the induced representation whose socle is $\mathrm{Symm}^{\underline{c}}$. I should say: we're doing this because it's the computation one needs to do when computing the full compact induction that Paskunas finds in $\text{c-ind}_{KZ}^G(\mathrm{Symm}^{\underline{c}})$. More precisely: let V be $\mathrm{Symm}^{\underline{c}}$ with c not on any boundary. Let I be the induced representation as above whose socle is V (the socle corresponds to $s = (0,0,\ldots,0)$ in the notation above). Note that I also has a unique irreducible quotient. Let X be the kernel of the map from I mod socle to the irreducible quotient. Then X has $2^f - 2$ J-H factors. Paskunas claims

$$0 \to \text{c-ind}_{KZ}^G X \to (\text{c-ind}_{KZ}^G V)/(T) \to P_V \to 0$$

where P_V is this H_0, the "Paskunas quotient" of the compact induction.

4.1 $f = 2$ Bardoe-Sin

s	λ_0	λ_1	a_0	a_1	r_0	r_1
$(0,0)$	c_0	c_1	0	0	c_0	c_1
$(0,1)$	c_0+p	c_1-1	c_0+1	0	$p-2-c_0$	c_1-1
$(1,0)$	c_0-1	c_1+p	0	c_1+1	c_0-1	$p-2-c_1$
$(1,1)$	c_0+p-1	c_1+p-1	c_0	c_1	$p-1-c_0$	$p-1-c_1$

4.2 $f = 3$ Bardoe-Sin

s	λ_0	λ_1	λ_2	a_0	a_1	a_2	r_0	r_1	r_2
$(0,0,0)$	c_0	c_1	c_2	0	0	0	c_0	c_1	c_2
$(1,0,0)$	c_0-1	c_1	c_2+p	0	0	c_2+1	c_0-1	c_1	$p-2-c_2$
$(0,1,0)$	c_0+p	c_1-1	c_2	c_0+1	0	0	$p-2-c_0$	c_1-1	c_2
$(0,0,1)$	c_0	c_1+p	c_2-1	0	c_1+1	0	c_0	$p-2-c_1$	c_2-1
$(0,1,1)$	c_0+p	c_1+p-1	c_2-1	c_0+1	c_1	0	$p-2-c_0$	$p-1-c_1$	c_2-1
$(1,0,1)$	c_0-1	c_1+p	c_2+p-1	0	c_1+1	c_2	c_0-1	$p-2-c_1$	$p-1-c_2$
$(1,1,0)$	c_0+p-1	c_1-1	c_2+p	c_0	0	c_2+1	$p-1-c_0$	c_1-1	$p-2-c_2$
$(1,1,1)$	c_0+p-1	c_1+p-1	c_2+p-1	c_0	c_1	c_2	$p-1-c_0$	$p-1-c_1$	$p-1-c_2$

5 "Companion" weights.

Switch notation. Fred's bs are helpful to have when thinking about his conjectures, but when messing around with combinatorics I've become more used to thinking of a weight as a pair $(\underline{a}, \underline{r}) = \det^a \mathrm{Symm}^r$.

The K-socle of a Paskunas irreducible has $I(1)$-invariants of dimension 2, so by a basic argument explains two Fred weights. Furthermore these weights are related: if one is associated to the character χ of the torus in $\mathrm{GL}_2(q)$ then the other is associated to χ^s. Unravelling this we see that if one of the weights corresponds to $(\underline{a}, \underline{r}) = (\underline{0}, (c_0, c_1, \ldots, c_{f-1}))$ then the other corresponds to $((c_0, c_1, \ldots, c_{f-1}), (p-1-c_0, p-1-c_1, \ldots, p-1-c_{f-1}))$. Note that for $f = 2$ such "Paskunas pairs" only show up in the reducible case.

More generally the weights that Fred et al predict naturally fall into pairs—given a weight $\det^{\underline{a}} \mathrm{Symm}^{\underline{r}}$ there is always a companion weight given by $\det^{\underline{a'}} \mathrm{Symm}^{\underline{r'}}$ with the property that the vector $r + r'$ is approximately $(p, p, p, \ldots, p)$, where by "approximately" I mean that one might get $p-1$ or $p-3$ instead of p. In the reducible case the construction of the companion weight is just given by complementing the set J; in the irreducible case it's given by complementing J'.

In general the principal series must I think be complements and their sum is $(p-3, p-3, \ldots, p-3)$. The Paskunas pair, as I said above, give $(p-1, p-1, \ldots, p-1)$. In the $f = 2$ case the four irreducible weights are in two pairs whose sum is $(p-1, p-3)$ and $(p-3, p-1)$. In the $f = 3$ reducible case we get the principal series $(p-3, p-3, p-3)$ and the three other pairs give $(p-1, p-1, p-3)$ and the cycles of this. In the $f = 3$ irreducible case we get the Paskunas pair which sum to $(p-1, p-1, p-1)$ and the other three pairs sum to $(p-3, p-3, p-1)$ and its cycles. In particular it seems to be a general principle that given a weight and its "companion", the sum mentions $p-1$s and $p-3$s, and there are an odd number of $p-1$s iff the weights show up in the irreducible case.

This lends credence to the idea that any irred rep of $\mathrm{GL}_2(\mathbf{Q}_{p^f})$ will show up either as a J-H factor of the rep associated to an irreducible mod p Galois rep, or a J-H factor of the rep associated to a reducible mod p Galois rep, but not both. One might be able to push these ideas as far as a purely combinatorial proof of this fact (assuming that a mod p correspondence exists and that Fred's conjectures are right).

6 Paskunas cycles.

In the $f = 2$ case the Palo Alto observation was the following. In the semisimple reducible case we could make a mod p local Langlands correspondence by attaching to it two principal series and a Paskunas. In the irreducible case we still have to do something, and it in fact must be something completely new. Here is the game one could play. Say Fred predicts $V := \mathrm{Symm}^{\underline{c}}$ for a Galois rep ρ in the irreducible case. Then we get a map from $S := \text{c-ind}_{KZ}^G(V)/(T)$ to $\pi(\rho)$. Now Paskunas has shown that $\text{c-ind}_{KZ}^G(V)/(T)$ contains a full compact induction from the direct sum of the two representations in the middle two lines of the $f = 2$ Bardoe-Sin table above, and the quotient is the Paskunas quotient. If the map from S to $\pi(\rho)$ had these full compact inductions in the kernel, then it would factor through the Paskunas quotient, but it can't do because the Paskunas quotient for V is isomorphic to the Paskunas quotient for some rep V' (a twist of $\mathrm{Symm}^{(\underline{p-1-c})}$) and hence there would be a non-trivial map from $\text{c-ind}_{KZ}^{G}(V')/(T)$ to $\pi(\rho)$ which would contradict Fred's conjectures. Hence one of the compact inductions is not in the kernel. But look at the tables: the two middle lines of the $f = 2$ Bardoe-Sin table only have one weight V_1 in common with the four weight in the $f = 2$ irreducible table. The one common weight is $(\underline{a}, \underline{r}) = ((0, c_1 + 1), (c_0 - 1, p - 2 - c_1))$. So by a similar argument we know that the map from S to $\pi(\rho)$ must be non-zero on $\text{c-ind}_{KZ}^{G}(V_1)$ and hence on $\text{c-ind}_{KZ}^{G}(V_1)/(T)$. Now do the argument again! This involves writing out the Bardoe-Sin table starting at V_1. On the other hand "pure thought" tells you that you don't need to do this because the new weight corresponds to $J' = \{3, 0\}$ and this weight is "formally isomorphic" to $\{0, 1\}$ after relabelling. So one will get each of the four weights showing up exactly once, one will get a subquotient of S which is again isomorphic to S, a mod p multiplicity one result would even give an explicit isomorphism between certain quotients of these copies of S, and I think that Matt even wonders whether somehow one can construct the representation from this data, or, perhaps more reasonably, from these ideas and a more careful analysis of the compact inductions.

In the $f = 3$ case the situation is as follows. In the irreducible case we have two weights explained by a Paskunas quotient, namely $J' = \{0, 2, 4\}$ and $\{1, 3, 5\}$, and we can try to run through the cycle argument as above. If one starts with the top line of the $f = 3$ irreducible table, with $J' = \{0, 1, 2\}$, then Paskunas tells us that in the compact induction mod T we see the full compact induction of something with 6 Jordan-Hoelder factors, namely those in the middle lines of the $f = 3$ Bardoe-Sin table. The first three, with $s = (1, 0, 0)$ etc, are the socle, and the other three are the cosocle. Again we get this miracle: in those six weights there is only one weight in common with the eight weights in the $f = 3$ irreducible picture, and it's the weight corresponding to $s = (1, 0, 0)$ in the Bardoe-Sin picture and $J' = \{5, 0, 1\}$ in the Fred weight picture. Again by pure thought we will now get all six weights showing up and this is enough to convince many of the experts that in the $f = 3$ irreducible case the six weights not explained by the Paskunas must be hence explained by precisely one irreducible whose K-socle will be the sum of Fred's six remaining weights.

The combinatorics of the $f = 3$ reducible case is a bit more subtle. We have the two principal series, and the six remaining weights correspond to twists where inertia has no fixed vectors (because both J and its complement are non-empty) so will be quotients of a compact induction mod T. Now we need to make another table to see what is going on because the one "basic" weight at the top of our table has gone—it's the principal series.

Let's take the weight V corresponding to $J = \{1, 2\}$, so it's the second one down. There will be a map from $\text{c-ind}_{KZ}^{G}(V)$ to $\pi(\rho)$ and it will factor through T but the Paskunas pair weight isn't on Fred's list so we'll have to be non-zero on the full compact induction. We need a list of the six weights involved in the full compact induction, and they can be obtained by taking our Bardoe-Sin $f = 3$ table, substituting $(p - 2 - c_0, c_1 + 1, c_2)$ for (c_0, c_1, c_2) and then twisting appropriately so that the top line of the resulting Bardoe-Sin table becomes the second line of the $f = 3$ Fred table. We get (omitting the λs which were only used to do the computation and we have no need for them now, we're just substituting):

s	a_0	a_1	a_2	r_0	r_1	r_2
$(0,0,0)$	c_0	$p-1$	$p-1$	$p-2-c_0$	c_1+1	c_2
$(1,0,0)$	c_0+1	$p-1$	c_2	$p-3-c_0$	c_1+1	$p-2-c_2$
$(0,1,0)$	0	0	0	c_0	c_1	c_2
$(0,0,1)$	c_0+1	c_1+1	0	$p-2-c_0$	$p-3-c_1$	c_2-1
$(0,1,1)$	0	c_1+1	0	c_0	$p-2-c_1$	c_2-1
$(1,0,1)$	c_0+1	c_1+1	c_2	$p-3-c_0$	$p-3-c_1$	$p-1-c_2$
$(1,1,0)$	$p-1$	$p-1$	c_2	c_0+1	c_1	$p-2-c_2$
$(1,1,1)$	$p-1$	c_1	c_2	c_0+1	$p-2-c_1$	$p-1-c_2$

Things are messier, which in some sense is good because in the non-semisimple reducible case one wants some more subtleties showing up. For Fred's conjectures to be true we must hope that the middle six lines in the table above contain something in common with the eight lines in the $f = 3$ reducible table. But in fact there are three weights in common! Two in the socle, including one of the principal series weights, and one in the "cosocle". The corresponding ss are $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$, so in fact all three of these weights are the J-H factors of a submodule of the full compact induction. The fact that one of the principal series occurs is some indication that there may be a non-split extension going on, which is good because we want to only predict a subset of weights sometimes. This might be worth thinking about more—Fred's predictions somehow form a lattice and it may be worth making some consistency checks here.

On the J side the subsets that show up are $\{0, 1, 2\}$ (the principal series), and $\{1\}$ and $\{0, 1\}$. So we can "move" from $J = \{1, 2\}$ to either $J = \{1\}$ or $J = \{0, 1\}$. The picture now is less clear. First I should say that by "switching characters of ρ", and hence replacing all Js by their complements, we can deduce what the picture will be if we had chosen $J = \{0\}$ as our base weight: we would have been able to switch to $J = \{0, 2\}$ or $J = \{2\}$, or to the other principal series (corresponding to J empty). Now by cycling we can get the general case without computing any more tables. This does seem to suggest that three of the weights are behaving differently to the other three and I am now becoming convinced that in the $f = 3$ reducible case there will be four representations, two principal series, two new things, each explaining 3 of Fred's weights, and each of the new things admits non-trivial exts with precisely one of the principal series somehow? One still needs more to explain what's going on though, doesn't one.

The naive combinatorial picture shows that we can either cycle as $\{1, 2\} \to \{0, 1\} \to \{0, 2\} \to \{1, 2\}$ or as $\{1, 2\} \to \{1\} \to \{0, 1\} \to \{0\} \to \{0, 2\} \to \{2\} \to \{1, 2\}$ giving a possible cycle of length 6, and there are also other possible asymmetric cycles which one might think cannot be ruled out by the combinatorics. On the other hand, because any cycle has length at least three, this shows that any non-principal series irreducible showing up in the $f = 3$ reducible case must explain at least three weights! So the only possibility is two irreducibles each explaining three, or one new irreducible explaining six.

7 Computations.

I wrote programs. I checked for $f \leq 10$ the following things.

1) The Paskunas irreducible weights show up in exactly one of the irreducible and reducible cases, and whichever one it's in depends on the parity of f.

2) More generally, if J is a subset of $\{0, 1, \ldots, f-1\}$ and J^c is the complementary subset then, in both the irreducible and the reducible case, there are weights V_J and V_{J^c}. If these are of the form $(\overline{a}, \overline{r})$ and $(\overline{a^c}, \overline{r^c})$. I checked that $r + r^c$ was always a vector comprising entirely of $p-1$s and $p-3$s, and that there were an odd number of $p-1$s iff we were in the irreducible case.

3) For $f > 1$ in the irreducible case, set $J' = \{0, 1, 2, \ldots, f-1\}$, and let's try and understand the irreducible representation explaining Fred's associated weight. By Paskunas avoidance the associated map from the compact induction mod T must be non-zero on the full compact induction. This object has $2^f - 2$ J-H factors. I checked that precisely one of them shows up on Fred's list! It's in the socle and it corresponds to $J' = \{-1, 0, 1, 2, \ldots, f-2\}$, so we're getting cycles of length $2f$ for all $f \leq 10$.

4) For $f = 4$ irreducible, the first irreducible case we don't conjecturally understand, Fred predicts 16 weights. The eight contiguous weights are in a cycle and probably form an irreducible, although I guess one can only prove that they are explained by an irreducible that might also explain other weights. The eight other weights are also in one cycle, and are generated by $J := \{0, 1, 3, 6\}$. By Paskunas avoidance, the 14 weights in the Bardoe-Sin module must have non-trivial intersection with Fred's 15 weights. But in fact the intersection has size 7! Four of the seven are contiguous, and the other three are $J + 2$, $J + 5$ and $J + 7$. So we could theoretically have eight irreducibles, each explaining one non-contiguous weight and all 8 contiguous weights, but this would contradict multiplicity one. More reasonable explanations are two irreducibles, each explaining 8, and three irreducibles, one explaining the contiguous eight and two more each

explaining four non-contiguous ones.

5) For $f \leq 9$ if a weight corresponds to a J which isn't a principal series or a Paskunas, then in both the reducible and irreducible cases, $J-1$ is in the allowed list of weights in the cycle, in the sense that the weight corresponding to $J-1=\{i-1 : iinJ\}$ is one of the Jordan-Hoelder factors in the full compact induction.

6) More generally, here is some structure that I just noticed in the cycle. There always appear to be 2^n-1 possible elements in the cycle. Furthermore n is always odd in the irreducible case, and always even in the reducible case. Are they coming from a submodule of the full induction?? Must check this later.

8 GL_n.

I briefly thought about computing Hecke algebras. If F is a finite extension of $\mathbf{Q}_p$, with integers $\mathcal{O}$ and residue field k and $\Gamma=\mathrm{GL}_n(k)$, then given an irreducible representation (ρ, V) of Γ over an algebraically closed field of characteristic p one can set $K=\mathrm{GL}_n(\mathcal{O})$, Z the centre of $G=\mathrm{GL}_n(F)$, and compactly induce ρ up from KZ to G. The resulting object has a Hecke algebra and one can ask what it is—and in particular whether is isomorphism class as an $\overline{\mathbf{F}}_p$-algebra depends on ρ or not!

The first thing to observe is that (letting ϖ be a uniformiser) $KZ\backslash G/K$ has a set of representatives $\mathrm{diag}(1,\varpi^{a_1},\varpi^{a_2},\ldots,\varpi^{a_{f-1}})$ where $0\leq a_1\leq a_2\leq\cdots\leq a_{f-1}$; this is just the structure theorem for modules over a DVR. In particular if for $1\leq i\leq f-1$ we set $\alpha_i=\mathrm{diag}(1,1,1,\ldots,1,\varpi,\varpi,\ldots,\varpi)$ with i 1s, then the monoid M generated by the α_i is a set of representatives for $KZ\backslash G/K$. Now to compute a basis for the Hecke algebra we need to compute the $\overline{\mathbf{F}}_p$-vector space of all functions in the compact induction that are in fact supported on $KZ\alpha KZ$, for any $\alpha\in M$. If ϕ is such a function then ϕ is determined by $\phi(\alpha)$, which must be an $\overline{\mathbf{F}}_p$-endomorphism of V with the property that $k_1\alpha=\alpha k_2$ implies $\rho(k_1)\phi(\alpha)=\phi(\alpha)\rho(k_2)$.

If V is 1-dimensional then it factors through det and we're fine: $k_1\alpha=\alpha k_2$ implies $\det(k_1)=\det(k_2)$ so $\phi(\alpha)$ can be anything non-zero. Hence the Hecke algebra has a basis $\{B_\alpha:\alpha\in M\}$ where B_α is the function supported on $KZ\alpha KZ$ sending α to the identity endomorphism of V.

If V is the standard representation then we can do some brute force computations for $n>2$. For $\alpha=1$ we deduce that $\phi(\alpha)$ had better commute with the action of ρ, so by Schur it's a scalar. For $\alpha=\alpha_i$ then letting $k_1=k_2$ be a matrix with two blocks, an i by i block and an $f-i$ by $f-i$ block, we deduce that $\phi(\alpha_i)$ had better commute with the action of all these things, and this easily implies (because $n>2!$) that $\phi(\alpha_i)$ had better be in block form too, with both the blocks scalars! Now letting k_1 be certain upper triangular unipotent matrices, with the non-zero off-diagonal entries being outside the "blocks", we deduce that the lower scalar had better be zero, so in fact the only way that α_i can act is via the matrix α_i mod ϖ, up to a constant, and again we've proved that there is only a 1-dimensional space of possibilities.

More generally if $\alpha\neq 1$ then α has a natural parabolic in Γ associated to it, coming from blocks corresponding to equalities amongst entries of α, and $\phi(\alpha)$ had better commute with the resulting Levi. This means that, for V the standard representation, $\phi(\alpha)$ is diagonal, with entries constant in the blocks corresponding to the Levi. Again using the unipotent radical of the parabolic one deduce that $\phi(\alpha)$ is acting via a multiple of the reduction mod ϖ of α. So again we have a basis for the Hecke algebra, and it's bijecting with the elements of the monoid.