Notes on Boutot-Carayol.

Kevin Buzzard

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1 Deligne.

Let K be a local non-archimedean field with integers $\mathcal{O}$, uniformiser π and residue field k of size q. Let C be the completion of an algebraic closure of K. Let Ω denote the set $C\backslash K = \mathbf{P}^1(C)\backslash\mathbf{P}^1(K)$. One can think of Ω as naturally corresponding to injective K-linear maps $K^2 \to C$, up to C^* homothety, the identification being that $\tau \in \Omega$ corresponds to the map $K^2 \to C$ sending (a,b) to $a\tau + b$. We will give Ω the structure of a rigid space over K, and a bit later on we will realise it as the generic fibre of a formal scheme over $\mathcal{O}$.

Recall the building I attached to $\mathrm{PGL}_2(K)$ is the graph with vertices the homothety classes of lattices in K^2, with an edge between $[M_1]$ and $[M_2]$ iff there are representatives M_1 and M_2 such that $\pi M_1 \subset M_2 \subset M_1$, the inclusions both being strict. The graph I is a tree with all vertices having valency $q+1$. Let $I_{\mathbf{R}}$ denote its real realisation; then $I_{\mathbf{R}}$ naturally parametrises (real-valued) norms on K^2 up to scaling; the vertex $[M]$ corresponds to the obvious norm with unit ball M and these norms interpolate nicely to also give norms for every point on the edges of $I_{\mathbf{R}}$.

Composing an injection $K^2 \to C$ with the usual norm on C we get a map $\lambda : \Omega \to I_{\mathbf{R}}$ whose image is $I_{\mathbf{Q}}$, the rational points (in the obvious sense) on $I_{\mathbf{R}}$. One checks that if s is a vertex of I then $\lambda^{-1}(s)$ is the C-points of the closed disc minus q open discs, and the pre-image of a (closed) edge $[s,s']$ is two such things glued together by an annulus (the annulus being the pre-image of the open edge). All of these things have natural structures as rigid spaces and one can glue them to give Ω the structure of a rigid space.

In fact one can do better. If $s = [M]$ then let $\mathbf{P}_s$ denote the projective space over $\mathcal{O}$ corresponding to M; it is non-canonically isomorphic to $\mathbf{P}^1$ over $\mathcal{O}$ but canonically isomorphic to $\mathbf{P}^1$ over K. Let Ω_s denote $\mathbf{P}_s$ with the k-rational points of the special fibre removed and let $\widehat{\Omega}_s$ denote its completion; this is an affine formal scheme whose generic fibre is naturally isomorphic to $\lambda^{-1}(s)$. One can do a similar thing with $\lambda^{-1}([s,s'])$ for $[s,s']$ a closed edge; we glue the results together and get a formal scheme $\widehat{\Omega}$ over $\mathcal{O}$ with generic fibre Ω.

Choose M and let $s = [M]$. Let F_s denote the functor from $\mathcal{O}$-algebras which are complete and separated with respect to the π-adic topology, to sets, sending R to the set of isomorphism classes of pairs $(\mathcal{L},\alpha)$ such that $\mathcal{L}$ is a free R-module of rank 1 and $\alpha : M \to \mathcal{L}$ is a homomorphism of $\mathcal{O}$-modules such that for all prime ideals x of R containing π the corresponding map $M/\pi M \to \mathcal{L}/x\mathcal{L}$ is injective.

Theorem 1. *F_s is representable by the formal scheme $\widehat{\Omega}_s$.*

There is a similar functor $F_{[s,s']}$ represented by $\widehat{\Omega}_{[s,s']}$.

2 Drinfel'd I (algebra).

We now write down a functor F on the category of $\mathcal{O}$-algebras B such that π is nilpotent on B, which is represented by $\widehat{\Omega}$. If B is such an algebra, set $B[\Pi] = B[X]/(X^2 - \pi)$, where Π is the image of X, and give it the $\mathbf{Z}/2\mathbf{Z}$-grading such that B has degree 0 and ΠB has degree 1. Let $S = \mathrm{Spec}(B)$. The functor F sends B to the set of isomorphism classes of quadruplets (η, T, u, r) where η is a constructible sheaf of flat $\mathcal{O}[\Pi]$-modules on S (under the Zariski topology) with a $\mathbf{Z}/2\mathbf{Z}$-grading, T is a $\mathbf{Z}/2\mathbf{Z}$-graded sheaf of $\mathcal{O}_S[\Pi]$-modules such that the graded pieces are both invertible sheaves on S, u is an $\mathcal{O}[\Pi]$-linear map $\eta \to T$ of degree 0 such that $u \otimes_{\mathcal{O}} \mathcal{O}_S : \eta \otimes_{\mathcal{O}} \mathcal{O}_S \to T$ is a surjection, r is an isomorphism $K^2 \to \eta_0 \otimes_{\mathcal{O}} K$, where here K^2 represents the constant sheaf on S, and such that (i) if S_i is the zero locus of the sheaf kernel of $\Pi : T_i \to T_{i+1}$ then $\eta_i|S_i$ is constant with fibre $\mathcal{O}^2$, (ii) for every geometric point x of S, if $T(x)$ is the fibre of T at s then the map $\eta_x/\Pi\eta_x \to T(x)/\Pi T(x)$ induced by u is injective, and finally $\Lambda^2\eta_i|S_i = \pi^{-i}(\Lambda^2(\Pi^i r\mathcal{O}^2))|S_i$ for both i.

Theorem 2. *F is representable by $\widehat{\Omega}$.*

The proof is to write down maps of functors $F_s \to F$ and $F_{[s,s']} \to F$ for all vertices s and edges $[s,s']$, to check they induce a map $\widehat{\Omega} \to F$, and then to check it's an isomorphism. For example, if $s = [M]$ and $\Lambda^2(M) = \pi^{-1}\mathcal{O}$ then the map $F_s \to F$ sends the pair $(\mathcal{L}, \alpha)$ to the data built by $\eta_0 = \eta_1 = M$, $T_0 = T_1 = \mathcal{L}$, $\Pi : \eta_0 \to \eta_1$ is the identity and $\Pi : \eta_1 \to \eta_0$ is multiplication by π, similarly for T_i, $u = \alpha$ and r is the isomorphism $K^2 \to M \otimes K$ induced by the embedding $M \to K^2$.

Note that $\mathrm{GL}_2(K)$ acts on Ω and $I_{\mathbf{R}}$, and also on $\widehat{\Omega}$, and also on F (although the action here is a bit messy), and everything commutes.

3 Drinfel'd II (p-divisible groups)

Why did Drinfel'd prove such a crappy technical theorem? Because it's the stepping-stone to a much more interesting one. Let K now have characteristic zero. The idea is that over $\mathcal{O}^{nr}$ the formal scheme $\widehat{\Omega}$, considered as a functor on $\mathcal{O}^{nr}$-algebras on which π is nilpotent, will be isomorphic to a functor related to p-divisible groups plus a quasi-isogeny. To prove such a thing all one has to do is to get from the p-divisible group to the linear algebra data of the previous section and this is done via the theory of Cartier modules.

Recall that a formal $\mathcal{O}$-module on an $\mathcal{O}$-algebra B is just a smooth formal group X/B with an $\mathcal{O}$-action such that the induced $\mathcal{O}$-action on the tangent space is the one coming from the B-action. Recall also that the category of formal $\mathcal{O}$-modules over B is equivalent to a linear algebra category, the category of Cartier $\mathcal{O}$-modules over B, that is modules over a monstrous non-commutative ring (a completion of something that looks like $W(B)[F, V]$ where W is Witt vectors) satisfying some axioms.

Now let $\mathcal{O}_D$ be the integers in a quaternion algebra over K, fix an embedding of $\mathcal{O}'$ into $\mathcal{O}_D$, where $\mathcal{O}'$ is the integers in the unramified quadratic extension of K, and choose $\Pi \in \mathcal{O}_D$ with $\Pi^2 = \pi$ and such that conjugation with Π induces the non-trivial automorphism of $\mathcal{O}'$.

Define a formal $\mathcal{O}_D$-module over an $\mathcal{O}$-algebra B to be a formal $\mathcal{O}$-module X equipped with an action of $\mathcal{O}_D$ and say that X is special if the induced action of $\mathcal{O}'$ on the tangent space of X makes it a free $B \otimes_{\mathcal{O}} \mathcal{O}'$-module of rank 1. The dimension of such a thing is 2 and they are equivalent to a certain linear algebra category (Cartier $\mathcal{O}$-modules plus some extra structure). Over $\overline{k}$, any special formal $\mathcal{O}_D$-module has height a multiple of 4, and there is one isogeny class of height 4 ones. Fix a certain element Φ in this isogeny class over $\overline{k}$.

The construction alluded to earlier in this section is the following. If B is an $\mathcal{O}$-algebra on which π is nilpotent, X is a special formal $\mathcal{O}_D$-module of height 4 over B, and ρ is a quasi-isogeny of height 0 from the base change of Φ to $B/\pi B$, to $X_{B/\pi B}$, then Drinfel'd constructs a quadruple $(\eta_X, T_X, u_X, r_{(X,\rho)})$, where T_X is just the tangent space of X, η_X and u_X come from the Cartier $\mathcal{O}$-module, and r is induced by the quasi-isogeny. One works hard (the statement is Theoreme II.8.2 of Boutot-Carayol but the proof is all of sections 9–12 of chapter II) to deduce that everything is a bijection and

Theorem 3. *On the category of $\mathcal{O}^{nr}$-algebras on which π is nilpotent, the functor sending B to the isomorphism classes of special formal $\mathcal{O}_D$-modules of height 4 over B equipped with a quasi-*

isogeny of height zero to our fixed such thing over $\overline{k}$, is representable by the formal $\widehat{\mathcal{O}}^{nr}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}$.

Note that as a consequence, there is a universal formal group over the base change of $\widehat{\Omega}$ to $\mathcal{O}^{nr}$, one can look at the points of exact order π^n on the generic fibre to get an interesting sequence of covers Σ_n of $\widehat{\Omega} \otimes_{\mathcal{O}} \widehat{K}^{nr}$, where $\widehat{K}^{nr}$ denotes the field of fractions of the completion of the strict henselisation of $\mathcal{O}$, and in particular the Σ_n are rigid spaces over $\widehat{K}^{nr}$ (and have models over K^{nr}, the field of fractions of $\mathcal{O}^{nr}$). One problem though is that the Σ_n are (or at least were in 1991) not known to be the generic fibre of some "natural" formal schemes, because one has technical problems with Drinfel'd bases in this situation. On the other hand a lot is known about the rigid spaces Σ_n, for example Carayol computed their cohomology to get a geometric realisation of both Langlands' and Jacquet-Langlands' correspondences between representations of $\mathrm{GL}_2(K)$, D^* and the Weil group W_K. Note however that Thoeorem 3 wasn't enough for Carayol—he needed Theorem 4, the Čerednik-Drinfel'd theorem, which is global (it's about Shimura curves over p-adic fields).

Important note: why is this local theorem true? Think about everything in terms of Dieudonne modules up to isogeny, and think about the case $\mathcal{O} = \mathbf{Z}_p$. We are looking at deformations of Φ and its Dieudonne module is the square of that of a supersingular elliptic curve over $\overline{k}$. If M_1 is the DM of a supersingular elliptic curve up to isogeny (so in particular it's a $\mathbf{Q}_p$-vector space) then $\mathrm{End}(M_1)$ is isomorphic the quaternion algebra D. Fix such an isomorphism i_1. Choose $\Pi \in D$ with $\Pi^2 = p$. Then we get another such isomorphism i_2 after conjugation by Π; let M_2 denote M_1 with this twisted action. Then $M := M_1 \oplus M_2$ is the DM of Φ; we twist to ensure the action is special.

Now to lift this DM is to choose a weakly admissible plane in M which is stable under the action of the quaternion algebra and such that the action on the plane is special. Choose a finite extension K of $W(\overline{k})$; tensor up to K and then choose your plane. The plane has an action of K and D so it has an action of $K \otimes D$ which is $M_2(K)$. Let L denote the unramified quadratic extension of $\mathbf{Q}_p$. Embed L into D so that conjugation by $\Pi \in D$ induces the non-trivial Galois automorphism of L. Now define M^0 to be the plane in $M \otimes K$ where the two actions of L coincide, and let $M^1 = \Pi M^0$.

To give the plane we're interested in is to give a line in M^0 which is an element of $\mathbf{P}^1(K)$. To ensure the line is weakly admissible one checks that we want to make sure that the line isn't defined over the ground field (as then the resulting line would be the tensoring up to K of a submodule of M defined over $W(\overline{k})$ which one can check contradicts weak admissibility). So we remove $\mathbf{P}^1(\mathbf{Q}_p)$ or $\mathbf{P}^1(W(\overline{k}))$ depending on whether we're working over $\mathbf{Q}_p$ or the field of fractions of $W(\overline{k})$.

4 Čerednik-Drinfel'd (global application)

I'll be brief. If Δ is an indefinite quaternion algebra over $\mathbf{Q}$ and p is a ramified prime and we look at a Shimura curve associated to Δ with no level structure at p, it's the solution to a moduli problem involving abelian surfaces with an action of $\mathcal{O}_\Delta$. This moduli problem can be extended to a moduli problem over $\mathbf{Z}_p$; one has to be a bit careful at points in characteristic p though—one imposes a condition that the induced action of $\mathbf{Z}_{p^2}$ on the Lie algebra of the tangent space is the direct sum of the two obvious eigenspaces—this is the analogue of the "special" condition in the previous section.

Representability of this functor over $\mathbf{Z}_p$ is standard, as long as one knows a lemma (Proposition 3.3 of Boutot-Carayol) about the existence of principal polarizations with certain properties, for abelian surfaces over nilpotent extensions of $\overline{k}$. Drinfel'd reduces this lemma to a question about p-divisible groups, verifies it for one p-divisible group over $\overline{k}$, and deduces the general result using his local representability result (although there are more direct proofs).

The result is a projective scheme S_U over $\mathbf{Z}_p$ with generic fibre equal to the Shimura curve. But even things like flatness are non-obvious, and an analysis of the special fibre as things stand might be tricky.

The big theorem is

Theorem 4. *The formal completion of S_U (as a formal scheme over $\mathbf{Z}_p$) is isomorphic to the quotient*

$$\mathrm{GL}_2(\mathbf{Q}_p)\backslash((\widehat{\Omega}\widehat{\otimes}\widehat{\mathbf{Z}}_p^{nr}) \times Z_U)$$

where Z_U is a certain discrete set depending on the level structure U.

Note that the p-adic upper half plane on the right is defined over $\mathbf{Z}_p^{nr}$ but the quotient is formally of finite type over $\mathbf{Z}_p$ (the action of $\mathrm{GL}_2(\mathbf{Q}_p)$ isn't defined over $\mathbf{Z}_p^{nr}$). One can manipulate the RHS until it becomes a finite union of quotients of $\widehat{\Omega} \otimes \mathbf{Z}_{p^n}$ so after extension to $\mathbf{Z}_{p^n}$ for some n, S_U becomes a finite union of "Mumford quotients" of the p-adic upper half plane.

As a consequence of this theorem, Kurihara's work tells us that S_U is flat, and also shows that the special fibre is reduced, only has ordinary double points as singularities, and even that the normalisations of the irreducible components of S_U are all rational curves!

When there is level structure at p one also gets something (Čerednik didn't get this, it's only because of Drinfel'd's reinterpretation that one gets anything):

Corollary 1. *The analytification of S_U is isomorphic to the quotient*

$$\mathrm{GL}_2(\mathbf{Q}_p)\backslash(\Sigma_n \times Z_{U^p})$$

where Σ_n is one of the covers of the rigid space Ω.

One deduces this easily from Theorem 4. Carayol uses Corollary 5 to compute the rigid-analytic cohomology of the spaces Σ_n.

The construction of the map is as follows: one starts by fixing an abelian surface over $\overline{k}$ with the usual properties, and one lets Φ denote its formal group. Then one considers all the algebraizations of Φ together with a level-U-structure, that is, all the pairs consisting of an abelian surface plus an isomorphism of the associated formal group with Φ and one checks that this space is just Z_U. This gives us a map from $\widehat{\Omega} \otimes \overline{k} \times Z_U$ to $(S_U)_{\overline{k}}$; one checks it's $\mathrm{GL}_2(\mathbf{Q}_p)$-invariant and induces an isomorphism on the quotient, so we're now done on the special fibre; one now uses Serre-Tate to get an isomorphism of the formal schemes.