Notes on $X_0(N)$.

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William Stein just explained to me (3 July 2002) what the Manin constant is. Along the way, we read some of a preprint called "the Manin constant, congruence primes, and the modular degree" by Amod and William. Along the way, we had to do a computation of how differentials on $X_0(N)$ change if one removes a singular point. This reminded me that I am always doing this kind of computation (it comes up in Coleman-Voloch, I think) and also I'm always trying to blow up the singularities of $X_0(N)$ to get a regular model. So here are some of the details of these things.

Firstly, not that it really matters for the below, note that we can use Katz-Mazur (KM) to compactify $Y_0(N)$ over $\mathbf{Z}$, that is, that we can apply section 8.6. The moduli problem $\Gamma_0(N)$ is relatively representable and finite over (Ell), and regular (all KM 6.6.1), and hence normal, and so the associated coarse moduli scheme $M(\Gamma_0(N))$ is normal, by KM 8.1.2. So one can use section 8.6 of KM to compactify it. The resulting compactification is called $M_0(N)$ by Amod and William, and Bas. Note that this isn't a fine moduli space and it's not necessarily a regular scheme, there are problems at supersingular points.

1 Differentials.

Let $M_0(N)^0$ denote the open of $M_0(N)$ comprising of the locus where the map down to $\mathrm{Spec}(\mathbf{Z})$ is smooth. I am not one hundred percent sure what this looks like in general, as I am so scared of these non-representable problems, but at the end of the day I think that you're doing something like throwing away all supersingular points in characteristics p dividing N exactly once, and all non-reduced fibres, that is, all middle components, in characteristics p dividing N more than once.

An example of all of this is $Y_0(2)$ over $\mathbf{Z}$; this is defined locally as $XY = 2^{12}$, as I recall. So the one non-smooth point will be defined by $X = Y = 2 = 0$.

The calculation which William and I did answered the following question: what is the difference between the local sections of Ω^1 on $M_0(2)$ and $M_0(2)^0$? More generally, if p divides N once, what's going on at the fibre at p and how do things change when one throws away the bad points?

Here's how we did it. Set $R = \mathbf{Z}[X,Y]/(XY - p^n)$, with $n \geq 1$ an integer. Locally this is the situation. Let's firstly compute the global sections of $\Omega^1_{R/\mathbf{Z}}$. This is easy, it's just $(RdX \oplus RdY)/(YdX - XdY)$. This ring is probably not a free R-module, it's bad at the maximal ideal (X, Y, p), although I don't prove this here. It must be bad here though, the map isn't smooth here.

Now let W be $\mathrm{Spec}(R)$ minus the non-smooth point. Note that $W = U \cup V$ where U and V are the loci defined by inverting X and Y respectively; any prime ideal containing both X and Y also contains p^n and hence p. An easy check shows that "the same" formulae give the differentials on U, V, and $U \cap V$; just replace R by the appropriate localisation. The sections of $\Omega^1_{R/\mathbf{Z}}$ on W can hence be computed using the sheaf axiom, once one gets ones head around this computation. Here's how we did it.

Work entirely within $A := R \otimes \mathbf{Q} = \mathbf{Q}[X, X^{-1}]$. Note that both X and Y are invertible in this ring, because p is. Now all these modules of differentials can be thought of as subsets of AdX. Let's write down $\mathbf{Z}$-bases for all of them. Over $\mathrm{Spec}(R)$ a $\mathbf{Z}$-basis is

$$\ldots, X^3dX, X^2dX, XdX, dX, YdX = -XdY, dY, YdY, Y^2dY, \ldots.$$

In terms of dX solely, the $\mathbf{Z}$-basis is

$$\ldots, X^2dX, XdX, dX, (p^n/X)dX, (p^n/X^2)dX, (p^{2n}/X^3)dX, (p^{3n}/X^4)dX, \ldots.$$

Now when we invert X, $R[1/X] = \mathbf{Z}[X, 1/X]$ and the differentials get much bigger: a $\mathbf{Z}$-basis is just $X^t dX$ for $t \in \mathbf{Z}$. And when we invert Y they get bigger in a "different direction"; a $\mathbf{Z}$-basis is things of the form $p^{-n(t+1)}X^t dX$ for $t \in \mathbf{Z}$. The sections on W are the intersection of these two things; the basis is $X^t dX$ for $t \geq -1$, and $p^{-n(t+1)}X^t dX$ for $t \leq -1$. This is very nearly the global sections of Ω^1 on all of $\mathrm{Spec}(R)$; the only difference between the two spaces of sections is at $X^{-1}dX$: over $\mathrm{Spec}(R)$ we only allowed $p^n X^{-1} dX$; over W we are allowed $X^{-1}dX$.

The upshot is that there's a map $\Omega^1(\mathrm{Spec}(R)) \to \Omega^1(W)$, and it's an injection of free $\mathbf{Z}$-modules, and the index is p^n, and the quotient is cyclic, generated by $X^{-1}dX = -Y^{-1}dY$, the "log pole" I guess. This differential clearly exists once one inverts X or Y, but it's $p^{-n}YdX$ and doesn't extend over the singularity.

2 Desingularisation: $uv = \pi^2$.

I'm sure this is all standard stuff. The idea is to blow up the singular points. Let's work in some generality: let's let $\mathcal{O}$ denote a complete DVR with uniformiser π, and let's consider a singularity of the form $\mathcal{O}[u,v]/(uv - \pi^2)$ first, and work our way up to more general π later. Note that this ring is not regular but it is an integral domain, and is, I believe, integrally closed.

Let's blow up the closed point. I am very naive about these matters. Here's the way I do it. Let $A = \mathcal{O}[u,v]/(uv - \pi^2)$. Let I denote the ideal (u, v, π). Let

B denote $A \oplus I \oplus I^2 \oplus \cdots$, considered as a graded ring. It would be nice to see a presentation of B. I think one is as follows: it's

$$A[U,V,\Pi]/(uV - Uv, u\Pi - U\pi, v\Pi - V\pi, uV - \pi\Pi, UV - \Pi^2).$$

It's hard to imagine anything else in the kernel. The first three relations are somehow "coming from projective space" and the last two are "coming from the relation".

The blow-up of A at I is just $\mathrm{Proj}(B)$, the homogeneous primes not containing all of U, V, Π. One can understand this better by covering $\mathrm{Proj}(B)$ by three affines gotten by inverting U, V and Π. Note that there's no need to invert Π though, as if Π is invertible then both U and V are too. So we can write $\mathrm{Proj}(B)$ as the union of two explicit affines. Here's one of them: the one gotten by inverting U.

The idea here is that $B_{(U)}$ is the degree 0 subring of B_U, the localisation of B at $\{1, U, U^2, \ldots\}$. Explicitly,

$$\begin{aligned} B_U &= A[U,U^{-1},V,\Pi]/(uV - Uv, u\Pi - U\pi, v\Pi - V\pi, uV - \pi\Pi, UV - \Pi^2) \\ &= \mathcal{O}[u,v,U,U^{-1},V,\Pi]/(uv - \pi^2, uV - Uv, u\Pi - U\pi, v\Pi - V\pi, uV - \pi\Pi, UV - \Pi^2). \end{aligned}$$

Now if I had any kind of geometric intuition at all I wouldn't have to do what I am about to do, which is to do algebra in the above ring. Firstly observe that U is invertible, so $UV = \Pi^2$ gives me V in terms of the other variables, and $Uv = \pi\Pi$ gives me v in terms of the other variables. So

$$\begin{aligned} B_U &= \mathcal{O}[u,U,U^{-1},\Pi]/(u\Pi\pi/U - \pi^2, u\Pi^2/U - \Pi\pi, u\Pi - U\pi, u\Pi^2/U - \pi\Pi) \\ &= \mathcal{O}[u,U,U^{-1},\Pi]/(u\Pi - U\pi). \end{aligned}$$

Now the degree 0 terms of this ring are

$$B_{(U)} = \mathcal{O}[u, \Pi/U]/(u(\Pi/U) - \pi)$$

and this is a regular ring! There's a natural map $A \to B_{(U)}$ sending u to u, and v to $\pi(\Pi/U)$. Note that this map is an injection.

Similarly, $B_{(V)} = \mathcal{O}[v, \Pi/V]/(v(\Pi/V) - \pi)$ and there's a natural map from A to this. The intersection is where both U and V are invertible; one can compute this for example by inverting V/U in $B_{(U)}$; this is tantamount to inverting Π/U and one sees that the intersection is $\mathcal{O}[\Pi/U, U/\Pi]$, considered as an A-module via the map sending u to $\pi U/\Pi$ and v to $\pi\Pi/U$.

One glues together $\mathrm{Spec}(B_{(U)}$ and $\mathrm{Spec}(B_{(V)})$ to get the blow-up of A at I. To understand this blow-up one can look at the fibres above various points of $\mathrm{Spec}(A)$. If m is a maximal ideal of A that isn't I, then general nonsense says that the fibre is just one point, because the blow-up doesn't change anything away from I. Remark: I just tried to see this explicitly in our case, but didn't find a general argument. One can just check on the special and on the generic fibre, perhaps this is one way to do it.

At I we compute the pre-images in both pieces. This just amounts to computing $\mathrm{Spec}(B_{(U)}/IB_{(U)})$ and similar for V. This is dead easy: $B_{(U)}/IB_{(U)} = \mathcal{O}[u, \Pi/U]/(u(\Pi/U) - \pi, u, \pi) = \mathcal{O}[\Pi/U]$ is the affine line. Similarly we get $\mathcal{O}[\Pi/V]$ in the other piece. Note that $\Pi/V = U/\Pi$. Looking in $B_{(UV)}$ we see that the image of I is (π) and so we are just creating projective 1-space over the residue field of $\mathcal{O}$. So we have desingularised our singularity and it's become a projective line.

In fact, an infinitely more interesting thing to do is to analyse the special fibre of the blow-up. This just involves modding everything out by π. Let k be the residue field of $\mathcal{O}$. Then A becomes $k[u, v]/(uv)$, $B_{(U)}$ becomes $k[u, \Pi/U]/(u(\Pi/U))$, $B_{(V)}$ is similar, and $B_{(UV)}$ becomes $k[(\Pi/U), (\Pi/U)^{-1}]$. The map $A \to B_{(U)}$ becomes the map sending u to u and v to 0, so it factors through $k[u]$. The map $B_{(U)}$ to $B_{(UV)}$ becomes the map sending u to 0 and Π/U to Π/U. Drawing some pictures shows that the special fibre of the blow-up is two affine lines and a projective line and the blow-down kills the projective line.

3 Desingularisation: $uv = \pi^n$

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Is this all just the same? Say $n \geq 3$.

Let A be $\mathcal{O}[u, v]/(uv - \pi^n)$ and let $I = (u, v, \pi)$. Using notation as in the previous section,

$$B = A[U, V, \Pi]/(uV - Uv, u\Pi - U\pi, v\Pi - V\pi, uV - \pi^{n-1}\Pi, UV - \pi^{n-2}\Pi^2),$$

and so now we see the difference: $\mathrm{Proj}(B)$ is now unfortunately only covered by the three affines gotten by inverting U, V and Π.

Now, analogous to before,

$$B_U = \mathcal{O}[u, v, U, U^{-1}, V, \Pi]/(uv-\pi^n, uV-Uv, u\Pi-U\pi, v\Pi-V\pi, uV-\pi^{n-1}\Pi, UV-\pi^{n-2}\Pi^2)$$

and now observing that $V = \pi^{n-2}\Pi^2/U$ and $v = uV/U$ gives us

$$B_U = \mathcal{O}[u, U, U^{-1}, \Pi]/(u^2\pi^{n-2}\Pi^2/U^2 - \pi^n, u\Pi - U\pi, u\pi^{n-2}\Pi^2/U - \pi^{n-1}\Pi)$$

and the second of these relations implies the other two. So

$$B_U = \mathcal{O}[u, U, U^{-1}, \Pi]/(u\Pi - U\pi)$$

and

$$B_{(U)} = \mathcal{O}[u, \Pi/U]/(u(\Pi/U) - \pi).$$

This is regular. The same sort of thing works for $B_{(V)}$. But the new piece of information, $B_{(\Pi)}$, looks like the following:

$$B_\Pi = \mathcal{O}[u, v, U, V, \Pi, \Pi^{-1}]/(uv-\pi^n, uV-Uv, u\Pi-U\pi, v\Pi-V\pi, uV-\pi^{n-1}\Pi, UV-\pi^{n-2}\Pi^2)$$

and writing $u = U\pi/\Pi$ and $v = V\pi/\Pi$ gives

$$B_\Pi = \mathcal{O}[U, V, \Pi, \Pi^{-1}]/(UV\pi^2 - \pi^n\Pi^2, VU\pi - \pi^{n-1}\Pi^2, UV - \pi^{n-2}\Pi^2),$$

the first two of these relations clearly being implied by the last. So

$$B_\Pi = \mathcal{O}[U, V, \Pi, \Pi^{-1}]/(UV - \pi^{n-2}\Pi^2)$$

and so

$$B_{(\Pi)} = \mathcal{O}[U/\Pi, V/\Pi]/((U/\Pi)(V/\Pi) - \pi^{n-2}).$$

So for $n \geq 3$, one has all the data one needs now. The special fibre is two affine lines and two projective lines, intersecting transversally: affine to projective to projective to affine. The outer two intersections are defined by equations of the form $xy = \pi$ and the intersection of the projective lines looks like $xy = \pi^{n-2}$. So one has to blow up $\lfloor n/2 \rfloor$ times and one ends up with $n - 1$ new projective lines, which was just what I wanted to see.