The L-group of $U(2)$.

Kevin Buzzard

April 26, 2012

Abstract

I thought I understood L-groups in 2008 until I realised that my thoughts on semi-direct products were too naive.

Last modified June 2008.

1 The definition.

Let $K/\mathbf{Q}$ be imaginary quadratic and define $G := U(2)$ to be the group over $\mathbf{Q}$ whose R-points are $\{x \in \mathrm{GL}_2(R \otimes_{\mathbf{Q}} K) : xx^{ct} = 1\}$ where c is complex conjugation on K and t is transpose. Then $G(\mathbf{R})$ is compact (it's the classical $U(2)$) and G satisfies Gross' equivalent conditions in Proposition 1.4 of his "algebraic modular forms" paper. Indeed G is an outer form of GL_2 so its centre will be a rank 1 torus, the diagonal matrices in G are $U(1) \times U(1)$ and the centre is $U(1)$ embedded diagonally. In particular, in Gross' notation $S = 1$ and $S' = 1$. Now $G(\mathbf{R})$ is compact so clearly contains no split torus of positive dimension (a closed subspace of a compact space is compact) and condition (5) of Gross' equivalent conditions is then easily verified.

2 The root data.

This is easy: you just work over the complexes. Fix once and for all an embedding $K \to \mathbf{C}$. Associated to this embedding is a natural map $\mathbf{C} \otimes_{\mathbf{Q}} K \to \mathbf{C}$ and hence an isomorphism of $G_{\mathbf{C}}$ with $\mathrm{GL}_2(\mathbf{C})$. The $U(1) \times U(1)$ (a maximal torus) in G becomes $\mathbf{C}^\times \times \mathbf{C}^\times$ over $\mathbf{C}$ and so $X^* = \mathbf{Z}^2$ (identifying (a,b) with the map $(x,y) \mapsto x^a y^b$) and the root is $(1,-1)$ and $X_* = \mathbf{Z}^2$ (identifying (c,d) with the map $z \mapsto (z^c, z^d)$ and the pairing is the dot product and the coroot is $(1,-1)$ too. "Inverse-transpose" is an outer automorphism of $\mathrm{GL}_2(\mathbf{C})$, but doesn't send the upper-triangular matrices to themselves. Better is the map $g \mapsto wg^{-t}w^{-1}$ with $w = \left(\begin{smallmatrix} 0 & \lambda \\ \mu & 0 \end{smallmatrix}\right)$ an invertible matrix. This outer automorphism fixes the upper triangular matrices. However we'd also like it to fix a "pinning", that is, the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and to rig this we need $\mu = -\lambda$, so we may as well set $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Now this outer automorphism sends a diagonal matrix $\mathrm{diag}(x,y)$ to the matrix $\mathrm{diag}(y^{-1}, x^{-1})$. Hence it corresponds to the automorphism of the (based) root data sending (a,b) to $(-b,-a)$ and (c,d) to $(-d,-c)$: note that both the root and the coroot are fixed by this. This is the only non-trivial automorphism of the based root datum because such an automorphism corresponds to $\alpha \in \mathrm{GL}_2(\mathbf{Z})$ with $\alpha(1,-1)^t = (1,-1)^t$ and $\alpha^t(1,-1)^t = (1,-1)^t$ and bashing these out gives $\alpha = \left(\begin{smallmatrix} 1+b & b \\ b & b+1 \end{smallmatrix}\right)$ and for the determinant to be ± 1 we'd better have $b = 0$ (the identity) or $b = -1$ (the non-trivial automorphism). The corresponding μ_G from the Galois group to the automorphisms of the based root datum factors through $\mathrm{Gal}(K/\mathbf{Q})$, which does indeed act non-trivially.

I learnt from Gross' article on algebraic modular forms that one also has an action of $\mathrm{Gal}(K/\mathbf{Q})$ on the Weyl group: this is because the Weyl group can be thought of as generated by the roots, and Galois acts on the roots. But because there is only one root, Galois acts trivially on the Weyl group. For general unitary groups it seems to act as conjugation by the longest root, but the Weyl group is abelian for $U(2)$.

3 The L-group (and something that isn't the L-group).

To define the L-group one has to actually really define a "lifting" of the automorphisms of the root data to the automorphisms of the dual group. One does this by fixing elements in the root spaces (over $\mathbf{C}$) and demanding that they get fixed too. In the GL_2 case we're self-dual, so we see that the lifting is just first inverse-transpose, and then conjugate by $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Hence the L-group of $U(2)$ is going to be the semi-direct product of $\mathrm{GL}_2(\mathbf{C})$ by $\mathrm{Gal}(K/\mathbf{Q})$, with complex conjugation acting on $\mathrm{GL}_2(\mathbf{C})$ via $g \mapsto wg^{-t}w^{-1}$. Our convention for the semidirect product will be that if σ is in the Galois group, then $(g, \sigma)(h, 1)$ will be $(g.\sigma(h), \sigma)$.

Now, to my surprise, Richard Taylor pointed out to me the following

Lemma 1. *Let τ_1 be the automorphism $g \mapsto g^{-t}$ of $\mathrm{GL}_2(\mathbf{C})$ and let τ_2 be the automorphism $g \mapsto wg^{-t}w^{-1}$ (with $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ as above). Then the semidirect products of $\mathrm{GL}_2(\mathbf{C})$ by $\mathbf{Z}/2\mathbf{Z}$ induced by τ_1 and τ_2 are* not *isomorphic as complex Lie groups!*

Proof. Let the semidirect products be L_1 and L_2. Any continuous isomorphism ϕ will induce an isomorphism on connected components of the identity, that is, a map $\mathrm{GL}_2(\mathbf{C}) \to \mathrm{GL}_2(\mathbf{C})$, which must then be either of the form $g \mapsto mgm^{-1}$ or $g \mapsto mg^{-t}m^{-1}$. Furthermore there will be $x \in \mathrm{GL}_2(\mathbf{C})$ such that $\phi(x\tau_1) = \tau_2$.

Let's now put these things together. Firstly $\phi(x\tau_1) = \tau_2$ implies that $x\tau_1$ has order 2 and hence $x\tau_1 x\tau_1 = 1$ so $xx^{-t} = 1$ so $x = x^t$ is symmetric.

Now $\phi(x\tau_1 g x\tau_1) = \tau_2\phi(g)\tau_2 = w\phi(g)^{-t}w^{-1}$ and hence

$$\phi(xg^{-t}x^{-1}) = w\phi(g)^{-t}w^{-1}$$

(recall that we've already proved that x is symmetric), and we have to unravel what this means in both cases.

(1) $\phi(g) = mgm^{-1}$. Then $mxg^{-t}x^{-1}m^{-1} = wm^{-t}g^{-t}m^t w^{-1}$ and hence $g^{-t} = Xg^{-t}X^{-1}$ for $X = x^{-1}m^{-1}wm^{-t}$. Hence X is a scalar, and $X = m^{-1}wm^{-t}x^{-1}$ (as $AB = \lambda$ implies $BA = \lambda$). But scalars are symmetric so $X = X^t = m^{-1}w^t m^{-t}x^{-1}$ and we deduce $w = w^t$, a contradiction.

(2) $\phi(g) = mg^{-t}m^{-1}$. Then $mx^{-1}gxm^{-1} = wm^{-t}gm^t w^{-1}$ and hence $Xg = gX$ for $X = m^t w^{-1} m x^{-1}$. So X is a scalar and $X = x^{-1}m^t w^{-1}m$ and $X = X^t = x^{-1}m^t w^{-t}m$ so $w = w^t$ again, which is also a contradiction. □

4 Conjugacy classes.

Langlands' interpretation of Satake's work involves consideration of the semisimple conjugacy classes in the $\mathbf{C}$-points of the L-group. One subtlety is that the classes can be in the non-identity component of the L-group, but you can only conjugate by things in the identity component. So if (x, τ_2) where F is p-adic field. I understand such things if $G(F)$ is isomorphic to $\mathrm{GL}_2(F)$. What about in the inert case? which have Galois component equal to Frobenius. In a more down-to-earth fashion I mean the following: if p is inert in K (the imag quad field) then the unramified representations of $U(2)(K_p)$ are supposed to biject with semisimple conjugacy classes in $\mathrm{GL}_2(\mathbf{C}).\tau_2$. Now $x\tau_2$ and $y\tau_2$ are conjugate iff they are related in the following way: $x\tau_2 = g^{-1}y\tau_2 g$, that is $x = g^{-1}ywg^{-t}w^{-1}$.

Before I embark on the explicit computation of these conjugacy classes, let me remark that if we had used τ_1 then the equivalence relation would be that y is related to $g^{-1}yg^{-t}$ and hence to gyg^t. Hence (setting $g = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we see that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is related to $\left(\begin{smallmatrix} 4 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. But one of these matrices is semisimple and the other isn't! Hence the semidirect product induced by τ_1 doesn't even have a notion of a semisimple conjugacy class!

OK so onto semisimple conjugacy classes in the L-group. The connected component is easy: $(x, 1)$ is called semisimple if x is, and in this case two semisimple elements are in the same class iff they're conjugatem which, for semisimple elements, is true iff they have the same char poly. If σ is an automorphism of a field k, and $x \in \mathrm{GL}_2(k)$ is semisimple, then x is group-)conjugate to $\sigma(x)$ iff σ fixes the char poly of x. If k is a field and K is an algebraic closure of k, say a conjugacy class in $\mathrm{GL}_2(K)$ is *defined over* k if for one (or equivalently, all) element x in the class, x is group-conjugate to all its Galois conjugates in $\mathrm{Gal}(K/k)$. The semisimple conjugacy classes defined over k are hence parametrised by the k-points of the variety $\mathrm{Spec}(\mathbf{Z}[t,d,d^{-1}])$. For GL_2 the notion of a conjugacy class being defined over k coincides with the notion of it containing a k-point, but this is not true for arbitrary connected reductive groups (Kottwitz gives an example: a unitary form of PGL_4).

Now let's consider the non-identity component.

Quick notes on the answer: the idea is that x is equivalent to $gxwg^tw^{-1}$ so if x is equiv to y then xw and yw represent the same quadratic form. If $xw = S + A$ with S symmetric and A alternating then $A = \lambda w$ and S is either non-degenerate, and hence can be put into the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, or can be put into the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ or 0. This leaves for xw the possibilities $\left(\begin{smallmatrix} 0 & 1-\lambda \\ 1+\lambda & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & -\lambda \\ \lambda & 0 \end{smallmatrix}\right)$ or λw, so x is either $\mathrm{diag}(\lambda - 1, 1 + \lambda)$ or non-semisimple or scalar; an explicit unilluminating calculation shows that anything equivalent to a non-semisimple matrix is non-semisimple; all the scalars are equivalent and in the final case the matrix is diagonal with distinct entries. Hence any twisted conjugacy class contains diagonal elements, and $\mathrm{diag}(a, b)$ is equivalent to $\mathrm{diag}(\mu a, \mu b)$ (an easy check) and to $\mathrm{diag}(b, a)$ [use w and possibly a scalar too] and that's it. So the semisimple conjugacy classes are naturally elements of a quotient of the torus (by the diagonal $\mathbf{G}_m$) modulo the usual switch, so are parametrised by $a/b+b/a$, the invariant of this setting, which is an element of the affine line.

I also checked that given a random E-point of the affine line, there was an E-point in the unitary group corresponding to this semisimple conj class. So rational conj classes contain rational points.