Harris-Taylor Local Langlands normalisation.

Kevin Buzzard

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Last modified 01/06/2004. How little I knew then!

I just want to write down the normalisation used by Harris-Taylor for the Local Langlands correspondence. Let K be a finite extension of $\mathbf{Q}_p$, let q denote the cardinality of the residue field. Let $|.| : K^\times \to \mathbf{C}^\times$ be the map sending a uniformiser to $1/q$. Let $\mathrm{Irr}(\mathrm{GL}_n(K))$ denote the irreducible admissible representations of $\mathrm{GL}_n(K)$ and let $\mathrm{WDRep}_n(W_K)$ denote the n-dimensionsal F-semi-simple representations of W_K, the Weil group of K.

Harris-Taylor let

$$\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$$

be the map sending π, a uniformiser, to a geometric Frobenius. They fix an isomorphism $\mathbf{C} = \overline{\mathbf{Q}}_l$. Let's fix a norm on W_K that's compatible with Art_K and the usual norm on $K^\times$; that is, define the norm $|g|$ of an element $g \in W_K$ by ensuring that the norm of a geometric Frobenius is $1/q$.

Their Local Langlands map is a bijection

$$\mathrm{rec}_K : \mathrm{Irr}(\mathrm{GL}_n(K)) \to \mathrm{WDRep}_n(W_K)$$

and they have another map r_l which takes an element π of $\mathrm{Irr}(\mathrm{GL}_n(K))$ to an l-adic Galois representation

$$r_l(\pi) : W_K \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l).$$

Note that in the simple case where $N = 0$ it is *not* true that $r_l(\pi) = \mathrm{rec}_K(\pi)$; there's a twist. Examples of this (I shall say more on these examples later): if $n = 1$ then

$$r_l(\pi) = \mathrm{rec}_K(\pi^\vee) = \mathrm{rec}_K(\pi)^{-1}$$

and if $n = 2$ then

$$\mathrm{rec}_K(\pi) = (r_l(\pi^\vee \otimes (|.|_K \circ \det)^{-1/2}), N)$$

for some N. Perhaps the point is that rec_K is the bijection that makes L and ϵ factors work out well, and r_l is the map that makes local and global Galois representations work well.

Let's work out some explicit examples of this (using other comments in Harris-Taylor which I won't record, to make things easier to read).

Say $n = 1$. Let's use Art_K to identify W_K^{ab} and $K^\times$. Say χ is an admissible representation of $K^\times$. Then

$$r_l(\chi) = \chi^{-1}$$

and

$$\mathrm{rec}_K(\chi) = \chi.$$

In particular, if χ is unramified and sends a uniformiser to α then $r_l(\chi)$ sends a geometric Frobenius to $1/\alpha$ and $\mathrm{rec}_K(\chi)$ sends a geometric Frobenius to α.

Now say $n = 2$ and χ_1, χ_2 are admissible characters of $K^\times$. Let π denote $I(\chi_1, \chi_2)$, the normalised induction from the usual character of the Borel, that is, locally constant $\phi : \mathrm{GL}_2(K) \to \overline{\mathbf{Q}}_l$ such that

$$\phi(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) g) = \chi_1(a)\chi_2(b)|a/b|_K \phi(g).$$

This is irreducible if $\chi_1/\chi_2 \neq |.|^{\pm 1}$; let's assume this. In this case, we have

$$r_l(\pi) = r_l(\chi_1) \otimes |.|^{-1/2} \oplus r_l(\chi_2) \otimes |.|^{-1/2}.$$

The simplest example of this: if the χ_i are unramified and χ_i sends a uniformiser to α_i then π is unramified principal series, the Hecke operators T and S defined using double cosets have eigenvalues $\sqrt{q}(\alpha_1 + \alpha_2)$ and $\alpha_1\alpha_2$ respectively, $r_l(\pi)$ sends a geometric Frobenius to a matrix with eigenvalues $\sqrt{q}/\alpha_1$ and $\sqrt{q}/\alpha_2$. To compute $\mathrm{rec}_K(\pi)$ I need to know $\pi^\vee$ and because we're using normalised induction we know that $\pi^\vee$ is the normalised induction of the dual of the representation of the Borel, so in particular it's $I(\chi_1^{-1}, \chi_2^{-1})$. Now an easy exercise shows that $I(\chi_1, \chi_2) \otimes (|.|_K \circ \det)^s \cong I(\chi_1|.|_K^s, \chi_2|.|_K^s)$ and hence if $\pi = I(\chi_1, \chi_2)$ then $\pi^\vee \otimes (|.| \circ \det)^{-1/2} = I(\psi_1, \psi_2)$ with ψ_i unramified sending a uniformiser to $\sqrt{q}/\alpha_i$. Hence $\mathrm{rec}_K(\pi)$ is a representation sending a geometric Frobenius to a matrix with eigenvalues α_1, α_2.