Notes on Langlands' proof of local Langlands in the abelian case.

Kevin Buzzard

April 26, 2012

Last modified 1st May 2009.

1 Summary

Langlands proves in his "Representations of abelian algebraic groups" paper, well, he proves several things, both local and global, and some compatibility statements, but I'm going to talk only about the local setting. So we'll let T be a torus over a local field F and we're after a bijection between continuous characters of $T(F)$ and Weil group representations into the L-group of T. Once we have this I'll talk about some other things.

2 The L-group.

Let me set up the notion of the L-group in the abelian setting. Let F denote a local field (either archimedean or non-archimedean), and let T be a commutative connected reductive group over F, that is, a torus over F. Let K denote a finite Galois extension of F over which T splits. Recall that the abelian group $L := \mathrm{Hom}(T_K, \mathbf{G}_{m,K})$ (the morphisms of algebraic groups—so L is the character group of T_K) determines T_K up to isomorphism, and one can recover $T_K(K) = T(K)$ from L thus: $T(K) = \mathrm{Hom}_{\mathbf{Z}}(L, K^\times)$, the dictionary being that $t \in T(K)$ corresponds to the map sending $\ell \in L$ to $t(\ell) := \ell(t)$.

We have T_K nailed now, but how to recover T? The extra structure on L that gives us T back is of course the Galois action. Set $\Gamma = \mathrm{Gal}(K/F)$. Now an element of L gives us a map $\ell : T(K) \to K^\times$. Galois acts on the left of both of these things, so for $\gamma \in \Gamma$ define $\ell.\gamma \in L$ by $(\ell.\gamma)(t) = \gamma^{-1}.\ell(\gamma.t)$. One can check that because one has used both γ and γ^{-1}, the resulting map $(\ell.\gamma) : T(K) \to K^\times$ is in fact the K-points of a map of algebraic groups over K, and this gives us a natural right action of Γ on L, which is well-known to determine T uniquely (and indeed even to lead to an antiequivalence of categories between tori over F that split over K and finite free $\mathbf{Z}$-modules with a right Γ-action). Furthermore, one can recover $T(K)$ with its left Γ-action as $\mathrm{Hom}_{\mathbf{Z}}(L, K^\times)$ with the following left action of Γ: if $t : L \to K^\times \in T(K)$ then $(\gamma.t)$ is defined by $(\gamma.t)(\ell) = \gamma(t(\ell.\gamma))$ and the proof that this is the correct action is an elementary unravelling that I'll omit (but which I did in rough!).

Let $\widehat{L}$ denote the dual lattice, with a natural left action of Γ, and define a right action of Γ on $\widehat{L}$ by inverting the action. This is highly implicit in Borel's article on L-functions in Corvallis! The inversion comes from the canonical identification of the automorphisms of a based root datum and the automorphisms of the dual datum: an automorphism of a based root datum will be automorphisms of both the character and cocharacter groups, but the abstract automorphism group will act on the character groups on one side and the cocharacter groups on the other side!

Now set $\widehat{T} = \mathrm{Hom}_{\mathbf{Z}}(\widehat{L}, \mathbf{C}^\times)$. Then $\widehat{T}$ is a complex Lie group (it's a product of copies of $\mathbf{C}^\times$). Now define a left action of Γ on $\widehat{T}$, but *be careful!* We don't have an action of Γ on $\mathbf{C}^\times$, and the definition is *not* the same as in the $T(K)$ case! We are invoking an equivalence of categories but not the one above, a different one. We are invoking the equivalence of categories between tori over $\mathbf{C}$ with a left action of Γ, and finite free abelian groups with a right action of Γ.

Explicitly, if $\tau \in \widehat{T}$ and $\phi \in \widehat{L}$ then define $(\gamma.\tau)(\phi) = \tau(\phi.\gamma) = \tau(\gamma^{-1}.\phi)$. Note that there is a genuine difference here: Γ acted on the left on $T(K)$ by semilinear maps which weren't in general induced by morphisms of algebraic groups $T(K) \to T(K)$, but it acts on the left on $\widehat{T}$ by complex analytic automorphisms and even by morphisms of algebraic groups over $\mathbf{C}$.

Finally, we can form the L-group ${}^LT = \widehat{T} \rtimes \Gamma$ (a complex Lie group, or an algebraic group). We need to have a concrete realisation of this group in a second, and we'll use the following: the underlying set of the L-group is $\widehat{T} \times \Gamma$ and multiplication is defined by $(t_1, g_1)(t_2, g_2) = (t_1(g_1 * t_2), g_1g_2)$ with $*$ the left action of Γ on $\widehat{T}$.

3 The Weil group.

Let $W_{K/F}$ denote the Weil group of the extension K/F, which is a canonical extension $0 \to K^\times \to W_{K/F} \to \Gamma \to 0$, but which is typically not the semi-direct product: the element of $H^2(\Gamma, K^\times)$ giving rise to the Weil group is the "canonical class" of local Class Field Theory. If $F = \mathbf{R}$ and $K \cong \mathbf{C}$ then the H^2 has order 2 and the Weil group corresponds to the non-trivial element.

It's not good form to define a group W_F, the "Weil group of F"; what one really should do is to choose an algebraic closure of F first. Note for example that the reals has two non-canonically isomorphic algebraic closures: the first is the complexes, and the second is a degree two extension of the reals which is non-canonically isomorphic to the complexes. Let's implicitly fix an alg closure of F.

Note that $W_{K/F}$ is actually naturally a quotient of W_F, if K is a subfield of the fixed algebraic closure of F; we know that W_K is a finite index normal subgroup of W_F, and $W_{F/K}$ is the quotient of W_F by the closure of the commutator subgroup of W_K.

Now let T be a torus over F and assume T splits over K, as before. We're interested in continuous homomorphisms $\phi : W_F \to {}^LT$ (note that both of these things surject onto Γ) such that

$$\begin{array}{ccc} W_F & \longrightarrow & \Gamma \\ \Big\downarrow \phi & & \Big\downarrow = \\ {}^LT & \longrightarrow & \Gamma \end{array}$$

commutes. Let me call such thing a "Weil group representation". Note that by commutativity of the square above, if we restrict a Weil group representation of W_F to W_K, the induced map $W_K \to {}^LT$ must have image in $\widehat{T}$ and hence factors through W_K^{ab}. Hence our original Weil group representation factors through $W_{K/F}$. If ϕ is a Weil group representation and we write $\phi(w) = f(w) \times g(w)$ with f in $\widehat{T}$ and $g(w) \in \Gamma$, then one checks easily that $f : W_{K/F} \to \widehat{T}$ is a continuous 1-cocycle (letting the Weil group act on $\widehat{T}$ via the Galois group) so ϕ gives us an element of $H^1_c(W_{K/F}, \widehat{T})$ (continuous cocycles, modulo coboundaries), and $H^1_c(W_{K/F}, \widehat{T})$ is by definition the set of equivalence classes of Weil group representations.

The "main theorem" that I'll talk about and around in this note is the following observation of Langlands: if K is local then the group $H^1_c(W_{K/F}, \widehat{T})$ canonically bijects with the continuous group homomorphisms $T(F) \to \mathbf{C}^\times$.

4 Construction of the bijection.

The actual details of the construction are tricky, because they involve inverting a map which is an isomorphism, but the fact that it's an isomorphism is not obvious.

The easiest way to explain the construction is to start with a representation of the Weil group W_F into the L-group and get a character of $T(F)$. So let's give ourselves a 1-cocycle $\rho : W_{K/F} \to \widehat{T}$, and an element $t \in T(F)$; we need to construct a complex number $\pi_\rho(t)$ such that $\pi_\rho : T(F) \to \mathbf{C}^\times$

is a continuous group homomorphism, and every continuous group homomorphism π is uniquely a π_ρ.

The dictionary is *almost* completely explicit, but we have to admit the fact that the natural trace map $H_1(W_{K/F},\widehat{L}) \to H_1(K^\times,\widehat{L})^\Gamma$ is an isomorphism. Note that this is group *homology*. Analogous to the fact that group cohomology H^1 is just group homomorphisms when the group acts trivially on the space, it turns out that $H_1(K^\times,\widehat{L})$ is just $(K^\times\otimes\widehat{L}) = \mathrm{Hom}_{\mathbf{Z}}(L,K^\times) = T(K)$, and that $H_1(K^\times,\widehat{L})^\Gamma$ is just the subgroup $T(F)$, but it seems to me that lifting an element of $T(F)$ to a 1-cycle on $W_{K/F}$ is tricky to do in practice, basically because the argument isn't completely formal, it relies on certain properties of the canonical classes in class field theory—in particular, if we replace $W_{K/F}$ by the semi-direct product of $K^\times$ and Γ then I don't think that the result remains true! I never followed this up, but the logic was that if $F=\mathbf{R}$ and $K=\mathbf{C}$ and $T=U(1)/\mathbf{R}$ then there seemed to be more maps from the semidirect product into the L-group than Weil group representations: there were reps of the semidirect product that didn't vanish on -1, but all reps of the L-group do.

This thorny issue notwithstanding, the rest of the actual *construction* is easy (although proving that it actually gives a bijection is harder). Here's the rest of the construction. Our task is, given a 1-cocycle $W_{K/F}\to\widehat{T}$, to construct a map $\pi : T(F)\to\mathbf{C}^\times$. So if we're also given an element in $T(F)$, we need to construct a complex number. Here's how Langlands does it. First lift the element of $T(F)$ to a 1-cycle in $H_1(W_{K/F},\widehat{L})$ (this is the thing I can't do explicitly, as I explained above). Now "as we all know" a 1-cycle can be thought of as a 1-chain, that is, a finite "sum" of pairs $(w_i,\hat{\lambda}_i)$, with some properties, modulo the 1-boundaries, which are 1-chains with some more restrictive properties. We evaluate the 1-cocycle on this 1-chain in the obvious way (the cocycle ϕ sends $(w,\hat{\lambda})$ to $\phi(w)\otimes\hat{\lambda}\in\widehat{T}\otimes_{\mathbf{Z}}\widehat{L})$, and we get an element of $\widehat{T}\otimes_{\mathbf{Z}}\widehat{L}$, and there's a canonical map from this to $\mathbf{C}^\times$ by definition of $\widehat{T}$. That's the complex number.

We'll come back to this, and in particular we'll say something about how we're going to get around the problem that the dictionary isn't actually explicit.

5 The construction in the split case.

This really is easy. In this case we can take $K=F$ and the issue of lifting the trace map disappears; we are in this situation if F is isomorphic to the complexes, for example. In this case Γ is trivial, and $W_{K/F}=K^\times$ (canonically: this is part of the data of a Weil group). Let's re-write the dictionary in this case. First we'll do some easy algebra.

Let W be an abelian topological group and let L be a finite free $\mathbf{Z}$-module.

{gp}

Lemma 1. *There are canonical bijections between the following two sets:*
(i) Π*, the set of continuous group homomorphisms* $\mathrm{Hom}_{\mathbf{Z}}(L,W)\to\mathbf{C}^\times$*, and*
(ii) R*, the set of continuous group homomorphisms* $W\to\mathrm{Hom}_{\mathbf{Z}}(\widehat{L},\mathbf{C}^\times)$*.*

Remark 2. Earlier versions of these notes also had (iii) the set of continuous bilinear maps $\widehat{L}\times W\to\mathbf{C}^\times$ (give $\widehat{L}$ the discrete topology and the left hand side the product topology). But I never used it really.

Remark 3. If V is a finite-dimensional vector space over a field then V is canonically isomorphic to its double-dual, and the canonical map from V to its double-dual is easy to write down, but to write down an inverse of this map one seems to need to write down a basis of V, or do something else "non-canonical", probably because the analogous result isn't true for V infinite-dimensional. The same is going on in the above lemma: we can write down a natural map in one direction between Π and R but to show that it's a bijection it seems that we have to choose a basis of L.

Proof. There's a natural map from Π to R: given π in Π we want a ρ, so given furthermore an element of W we need to find a map $\widehat{L}\to\mathbf{C}^\times$, and so if we're also given $\hat{\lambda}\in\widehat{L}$, we want a complex number! To get it, we evaluate π on the map $\ell\mapsto w^{\hat{\lambda}(\ell)}$, and this gives us a complex number.

That's it! Call the resulting map ρ_π. The hard part is going the other way, which I don't know how to do "canonically". But here's a trick.

If $\Lambda \cong \mathbf{Z}^n$ then both Π and R are non-canonically isomorphic to H^n, with H the set of continuous group homomorphisms $W \to \mathbf{C}^\times$. If (f_i) is a $\mathbf{Z}$-basis for $\widehat{L}$ then here's the dictionary in both cases: given an element π of Π define $\phi_i(w) = \pi(\lambda \mapsto w^{f_i(\lambda)})$ and given an element ρ of R define $\phi_i(w) = \rho(w)(f_i)$. One checks easily that a change of basis $f_i \to f_i'$ changes the ϕ_i to ϕ_i' with exactly the same transformation matrix, so the induced isomorphism between Π and R is independent of choice of basis. Furthermore it's elementary to check (I did it but won't write down the details) that the ϕ_i associated to π and ρ_π are the same, so this shows that the map we wrote down is an isomorphism.

□

As I said, I can't seem to go from R back to Π without picking a basis, which is a shame because this bijection *is* local Langlands in the split case! If $W = K^\times$ and T is a torus over K, split over K, with character group L, then Π is the continuous group homs $T(K) \to \mathbf{C}^\times$ and R is the continuous group homs $K^\times \to \widehat{T}$, and furthermore the canonical map we've just explained is Langlands' bijection: when you unravel Langlands' bijection this is what you get.

6 Tangent spaces.

Now say F is isomorphic to the reals, so K, a finite extension of F, is a finite-dimensional real vector space. Then spaces like $\mathrm{Hom}_{\mathbf{Z}}(L, K^\times)$ and things all become real manifolds, and they all have tangent spaces. Let me take the tangent space of the preceding lemma.

{alg}

Lemma 4. *There are canonical* $\mathbf{R}$*-linear isomorphisms between the following two sets:*
(i) The set T_Π *of* $\mathbf{R}$*-linear maps* $\mathrm{Hom}_{\mathbf{Z}}(L, K) \to \mathbf{C}$
(ii) The set T_R *of* $\mathbf{R}$*-linear maps* $K \to \mathrm{Hom}_{\mathbf{Z}}(\widehat{L}, \mathbf{C})$.

Proof. Again there's a natural map from T_Π to T_R: given π in T_Π define ρ_π in T_R by $((\rho_\pi)(k))(\hat{\lambda}) = \pi(\ell \mapsto \lambda(\ell)k)$. Again if one chooses an isomorphism $L \cong \mathbf{Z}^n$ then one sees that both T_Π and T_R are isomorphic to $\mathrm{Hom}_{\mathbf{R}}(K, \mathbf{C})^n$ as real vector spaces and that the map I just described is an injection and hence an isomorphism. □

The reason I'm interested in tangent spaces is that when $F = K$ is Archimedian, there are maps $\Pi \to T_\Pi$ and $R \to T_R$ given by differentiation, and the derivative of the natural map in Lemma 1 is the natural map in Lemma 4. We get elements of T_Π by looking at the representation of the Lie algebra of $T(K)$ associated to a representation of $T(K)$, and elements of T_R from the derivative of the Weil group representation, and so local Langlands in the case where $K = F$ is archimedian and the group is a split torus induces a natural identification between those two spaces and what we see is that it's the "obvious" one.

7 The non-split case: some algebra.

Let me beef up those last lemmas a bit. Let's say that there's a finite group Γ which acts on the right on L, and on the left on $\widehat{L}$, and which also acts continuously on the left on W. Define right Γ-actions on the sets Π and R of Lemma 1 above, thus. Say $\gamma \in \Gamma$. If $\pi \in \Pi$ then define $(\pi.\gamma)(t)$ to be $\pi(\gamma.t)$ where γ acts on the left on $\mathrm{Hom}_{\mathbf{Z}}(L, W)$ by $(\gamma.t)(\ell) = \gamma(t(\ell.\gamma))$. And if $\rho \in R$ define $(\rho.\gamma)(w)(\hat{\lambda})) = \rho(\gamma.w)(\gamma.\hat{\lambda})$. One checks easily that these are actions. The claim is that the canonical bijection $\Pi = R$ is Γ-equivariant, and to check this all I have to do is to check that the natural map I gave above from Π to R commutes with Γ. And it does: this is a tedious check, which I did but I'll omit.

Let's also put an action on Lemma 4, assuming that K contains $\mathbf{R}$ and that the Γ-action on K is $\mathbf{R}$-linear; then we get Γ actions on both T_Π and T_R using basically the same definition, and the natural map between them commutes with the action.

8 Local Langlands: An explicit fragment of the general case.

I never worked out how to explicitly realise the isomorphism $T(F) = H_1(W_{K/F}, \widehat{L})$; the natural map went in the other direction and I couldn't see a natural inverse. But let me explain a trick which lets me see what's going on infinitesimally without ever having to invert this map.

There is a natural map $T(K) \to H_1(W_{K/F}, \widehat{L})$, namely corestriction (recall that $T(K) = H_1(K^\times, \widehat{L})$, and corestriction corresponds to the "obvious" thing on chains; the 1-chain in $H_1(W_{K/F}, \widehat{L})$ corresponding to an element of $T(K)$ vanishes, by definition, off $K^\times$).

Note that $\widehat{L}$ has a left action of $W_{K/F}$ (via Galois, where we note that usually we think of it as a right action so we have to invert to get the left action) and hence $H_1(K^\times, \widehat{L})$ has an induced action of Γ, by some formal argument.

Lemma 5. *The canonical isomorphism $H_1(K^\times, \widehat{L}) = T(K)$ commutes with the action of Γ.*

Proof. One can compute explictly on the chain level. If $\phi \in \widehat{L}$ and $k \in K^\times$ then $(k)\phi$ is a 1-chain, and the associated element of $T(K) = \mathrm{Hom}(L, K^\times)$ sends λ to $k^{\phi(\lambda)}$. If $w \in W_{K/F}$ then w sends $(k)\phi$ to $(wkw^{-1})(w\phi)$ and if w corresponds to γ in the Galois group then this is $(\gamma.k)(\gamma.\phi)$. The associated element of $T(K)$ sends λ to $(\gamma k)^{\phi(\lambda.\gamma)}$ so we have to check that this equals γ of the map $L \to K^\times$ above, and we worked out earlier how this acted: γ of $\lambda \mapsto k^{\phi(\lambda)}$ is the map $\lambda \mapsto \gamma(k^{\phi(\lambda.\gamma)})$ and lo and behold we're done. □

As a consequence we deduce that Langlands is right when he says that the Γ-invariants in $H_1(K^\times, \widehat{L})$ are just $T(F)$.

Furthermore we can check that the composite map $T(K) \to H_1(W_{K/F}, \widehat{L}) \to T(F)$ of corestriction and trace is the norm map. We check on 1-chains, as usual. If $k \in K^\times$ and $\phi \in \widehat{L}$ then $(k)\phi$ is a 1-chain in $H_1(K^\times, \widehat{L})$ and if we apply corestriction and then trace we get $\sum_\gamma (\gamma.k)(\gamma.\phi)$, so it's just the trace map on $H_1(K^\times, \widehat{L})$ and lo and behold we just checked that the identification of this map with $T(K)$ was Galois-invariant, so we're home.

Hence on elements of $T(F)$ which are norms from $T(K)$, and this does not include every element but it will include an open neighbourhood of the identity if $F = \mathbf{R}$, Langlands' correspondence will be easy to write down explicitly. So let's do it! Given a Weil representation $\phi : W_{K/F} \to {}^LT$, restrict to $\phi : K^\times \to \widehat{T}$. By the split case of local Langlands we get a map $T(K) \to \mathbf{C}^\times$ and this map factors through the norm map $T(K) \to T(F)$ (by Langlands' theorem! This seems tricky to verify directly!); we do however *not* get an induced map $T(F) \to \mathbf{C}^\times$ with this recipe because the norm map isn't surjective, but the character that Langlands associates to the Weil representation will at least be given by this recipe on the image of the norm map. It's not surprising that we can't finish the job here: we only defined π on $N(T(K))$ instead of all of $T(F)$, but the only data we used was $\rho|K^\times$ rather than all of ρ.

Let me summarise this last paragraph with a commutative diagram. Let F be a local field, let T/F be a torus, and let K be finite Galois extension of F where the torus splits. Given a map $\rho : W_F \to {}^LT$, Langlands associates $LL(\rho) : T(F) \to \mathbf{C}^\times$. But we can also restrict ρ to $K^\times$ and this is a map $W_K \to \widehat{T}$ and Langlands associates to this a map $LL(\rho|K^\times) : T(K) \to \mathbf{C}^\times$. My assertion, which comes directly from Langlands' construction of his bijection in this case, is that the following diagram commutes:

$$\begin{array}{ccc} T(K) & \xrightarrow{LL(\rho|K^\times)} & \mathbf{C}^\times \\ \downarrow N & & \downarrow = \\ T(F) & \xrightarrow{LL(\rho)} & \mathbf{C}^\times \end{array}$$

Note however that the top line does not determine the bottom line, because the norm map isn't in general surjective, and this isn't surprising, because the top line doesn't use all of the information about ρ.

9 Local Langlands infinitesimally.

By this I mean: again set $F = \mathbf{R}$. Let K be an algebraic closure of $\mathbf{R}$, and let T be any torus over F. I've described how Langlands gives a bijection between ρs and πs with the usual notation. In the previous section I noted that π restricted to $N(T(K))$ only depends on ρ restricted to $K^\times$. The neat thing is that in the archimedian case the image of the norm map contains a neighbourhood of the identity! Hence there's a dictionary on tangent spaces, and we've seen these things "explicitly" so we should be able to work out the dictionary.

OK so let's go. Let's say we're given a Weil representation $\rho : W_{K/F} \to {}^L T$. Restrict it to $K^\times$ and we get a map $\rho_K : K^\times \to \hat{T}$. Furthermore, this map has the property that it came from a Weil group representation and hence $\rho_K(\gamma.w) = \gamma.(\rho_K(w))$ for $w \in K^\times$. Let's see how this fits into our linear algebra world! Unravelling the above statement we see that for $\hat{\lambda} \in \widehat{L}$ we have $(\rho_K(\gamma w))(\hat{\lambda}) = (\rho_K(w))(\gamma^{-1}\hat{\lambda})$. And, by definition of the right action of Γ on R, we see that this statement is just the statement that ρ_K is Γ-invariant!

Remark 6. It is not the case that a Γ-invariant ρ always extends to a representation of the Weil group! We'll see this in the case of $U(1)$ later. If the Weil group were a semidirect product then representations would always extend and local Langlands would be wrong.

Now let's consider things on the π side. Say π is a map $T(F) \to \mathbf{C}^\times$. Consider the induced map $\pi_K : T(K) \to \mathbf{C}^\times$ that one gets by composing with $N : T(K) \to T(F)$. This map has perhaps lost a little of π's information, but that's OK. Furthermore, $\pi_K(\gamma t) = \pi_K(t)$, because these elements have the same norms. Now $\pi_K \in \Pi$ and we instantly see that it's Γ-invariant. Note however that not every Γ-invariant π_K factors through the norm map! We'll see a counterexample for $U(1)$ later on.

But on tangent spaces, all these subtleties disappear. We have just seen that for π and ρ associated via local Langlands for T/F, the associated $\pi_K \in \Pi$ and $\rho_K \in R$ are G-invariant, and are associated by local Langlands for T_K/K. Let's carefully unravel what we have.

Say ρ and π are related via local Langlands for $T/\mathbf{R}$. Then $\rho_K \in R$ and $\pi_K \in \Pi$ are related by local Langlands for T/K. But there's a subtlety. The derivative of π on $T(F)$ is a map from the Γ-invariant elements of the tangent space of $T(K)$, to $\mathbf{C}$. So it's a map $\mathrm{Hom}_{\mathbf{Z}}(L, K)^\Gamma \to \mathbf{C}$. We do *not* get this map by restricting t_{π_K}. We get it by norming it! This, vaguely speaking, is why to properly understand the dictionary of local Langlands you have to not fix an isomorphism $K = \mathbf{C}$. If you did then you might forget to norm down, an action which in some sense involves all the isomorphisms $K \to \mathbf{C}$ at once.

10 Examples in the arch. case.

Let F be the reals or an algebraic closure of the reals, and let T be a torus over F. We need to choose K splitting T, so we can sometimes choose $K = F$ and we can always choose $K = \overline{F}$. Given a ρ let's build the corresponding π.

10.1 $T = \mathrm{GL}_1$ over $F = \mathbf{R}$.

Let G be $\mathrm{GL}_1/\mathbf{R}$. Then π is a map $\mathbf{R}^\times \to \mathbf{C}^\times$ and ρ is a map $W_{\mathbf{R}} \to \mathbf{C}^\times$. If we set $K = \mathbf{R}$ then $L = \widehat{L} = \mathbf{Z}$ and ρ is a representation of $W_{\mathbf{R}/\mathbf{R}} = \mathbf{R}^\times$ and so ρ and π are "the same thing"! The bijection is the obvious one.

If however we are a bit bonkers and let K be an algebraic closure of $\mathbf{R}$ then ρ is supposed to be a map $W_{K/\mathbf{R}} \to \mathbf{C}^\times$—but this factors through the abelianisation of $W_{K/\mathbf{R}}$ which is $\mathbf{R}^\times$ again. Note that if we restrict $\rho : W_{K/F} \to \mathbf{C}^\times$ to $K^\times$ then we get a map $\rho_K : K^\times \to \mathbf{C}^\times$ with $\rho_K(ck) = \rho_K(k)$ and π_K is a map $K^\times \to \mathbf{C}^\times$ with $\pi_K(k) = \pi_K(ck)$. Note that a necessary and sufficient condition for a continuous group homomorphism $K^\times \to \mathbf{C}^\times$ to factor through $N : K^\times \to F^\times$ is that it vanishes on the unit circle, which it will do because any z in the unit circle can be written k/ck. The arguments above show us that the induced map $K^\times \to W_F \to W_F^{ab} = F^\times$ should be the norm map, which indeed it is.

Thought of in this way, t_{π_K} is an $\mathbf{R}$-linear map $K \to \mathbf{C}$ and t_{ρ_K} is an $\mathbf{R}$-linear map $K \to \mathbf{C}$ and t_π is an $\mathbf{R}$-linear map $F \to \mathbf{C}$, that is, a complex number s, and $t_{\pi_K} = t_{\rho_K}$ is the map $K \to \mathbf{C}$ sending t to $st + sct$.

10.2 $F = \overline{\mathbf{R}}$ and $T = \mathrm{GL}_1$.

Set $K = F = \overline{\mathbf{R}}$ (the only choice). Then $L = \mathbf{Z}$ and ρ and π are both maps $K^\times \to \mathbf{C}^\times$, and local Langlands demands that they coincide. Note that there are quite a lot of such maps! If σ and τ are the two isomorphisms $K \to \mathbf{C}$ then ρ and π are of the form $k \mapsto \sigma(k)^a \tau(k)^b$ with $a, b \in \mathbf{C}$ and $a - b \in \mathbf{Z}$.

10.3 $F = \mathbf{R}$ and $T = U(1)$.

Now we have to set $K = \overline{F}$. Define $G := U(1)$ thus: $G(A) = \{x \in A \otimes_F K : x\overline{x} = 1\}$ where $\overline{*}$ is the A-linear map on $A \otimes_F K$ which is the non-trivial Galois element on K. Now there are two ways of identifying G/K with GL_1; let's choose one. Then $L = \mathbf{Z}$. Of course the action of Γ isn't trivial this time: c sends n to $-n$, so $\widehat{T}$ is just $\mathbf{C}^\times$ with $cz = 1/z$. A Weil group representation is $\rho : W_F \to {}^LT$ and this restricts to a map $\rho_K : K^\times \to \widehat{T}$ with $\rho_K(cz) = 1/\rho_K(z)$. Note however that not every such ρ_K extends to a Weil group representation! If ρ is a Weil group representation then $\rho(j)$ will be in the non-trivial component of the L-group, but it's easily checked that everything in the non-trivial component has order 2, so $\rho(j)$ has order 2, so j^2 is in the kernel of ρ, so $\rho_K(-1) = 1$. There are however Galois-stable maps $K \to \mathbf{C}^\times$ (for example the one sending $re^{i\theta}$ to $e^{i\theta}$ which don't sent -1 to -1. This is one reason why the Weil group had better not be the semidirect product! In fact a Galois-stable map had better send $F^\times$ to ± 1 and because the image of j has order 2 a Weil group representation had better be trivial on all of $F^\times$ and, if we fix an isomorphism $K = \mathbf{C}$, had better send $re^{i\theta}$ to $e^{2in\theta}$ for some integer n, so if we fix $K = \mathbf{C}$ then it sends z to $(z/\overline{z})^n$.

Now what about π? Well π is a map $T(F) \to \mathbf{C}^\times$, and $T(F)$ is non-canonically the unit circle, so π is non-canonically an integer, but this deserves to be done more carefully.

11 Restriction of scalars.

See section 8.4 of Borel Corvallis. In our case it's the observation that if $F = \mathbf{R}$ and $K = \overline{F}$ then $\mathbf{C}^\times$ is a module for $K^\times$ (with the trivial action) and inducing this module up to $W_{K/\mathbf{R}}$ gives a group $(\mathbf{C}^\times)^2$ with Galois switching the two components, so by Shapiro's Lemma we have $H^1(K^\times, \mathbf{C}^\times) = H^1(W_{K/\mathbf{R}}, \hat{T})$ with $T = \mathrm{Res}_{K/\mathbf{R}}\,\mathrm{GL}_1$.

12 Deligne's S.

Let $F = \mathbf{R}$, $K = \overline{F}$, and $T = \mathrm{Res}_{K/F}\,\mathrm{GL}_1$. Then $T(F) = K^\times$, $L \cong \mathbf{Z}^2$ with Galois action $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so the dual of L is isomorphic to L and the L-group ${}^LT$ is $(\mathbf{C}^\times)^2$ semidirect Galois, with Galois switching the two coordinates. The idea is that given a map $W_{K/F} \to {}^LT$, the restriction to both $K^\times$ (on the left) and "the top left hand component" (on the right) gives a map $K^\times \to \mathbf{C}^\times$. This is delicate. Somehow the point seems to be that the dual group $\hat{T}$ is induced from $C^\times$, and knows "which $\mathbf{C}^\times$ it came from"?? i.e. $\hat{T}$ really is "$\mathbf{C}^\times xc\mathbf{C}^\times$ with c complex conjugation"? I am a bit muddled by this but it might really be how it's working.

13 Some once useful remarks.

I tried to check that the explicit dictionary I have for local Langlands gives me a map $T(K) \to \mathbf{C}^\times$ that factors through the norm! It wasn't formally true! Given a Weil rep it restricts to a rep ρ of $K^\times$ such that $\rho \circ c = c \circ \rho$ as it were, but this is not a sufficient condition for a rep ρ of $K^\times$ to come from a rep of the Weil group! Take for example the case $T = U(1)$. Then $L = \mathbf{Z}$ with c acting as -1, so $\widehat{T}$ is $\mathbf{C}^\times$ with c sending z to $1/z$. Now here's a ρ that commutes with c: send $re^{i\theta} \in K^\times$ to $e^{i\theta}$. But this does *not* extend to a map of the Weil group! Because j will have to go to something in the non-trivial component of the L-group, and everything there has order exactly 2, and so j^2 goes to 1, but $j^2 = -1 \in K$ and this doesn't.

Borel says: if T is a torus over F and K is isomorphic to the complexes then to give $\pi : K^\times \to \widehat{T}$ is to give a pair (λ, μ) of elements of $X^*(T) \otimes \mathbf{C}$, indexed by isomorphisms $K = \mathbf{C}$, subject to $\lambda - \mu \in X^*(T)$.

He also says that for G a torus over $\mathbf{C}$, group homs $G \to \mathbf{C}^\times$ are (λ, μ) in $X^*(G) \otimes_{\mathbf{C}}$ with difference in $X^*(G)$, and to tive $\mathbf{C}^\times \to G$ is to give $(\mu, \nu) \in X_*(G) \otimes \mathbf{C}$ with difference in $X_*(G)$. He also says that the Lie algebra of a complex torus G is just $X_*(G) \otimes \mathbf{C}$ and the exponential map is an isomorphism $X_*(G) \otimes \mathbf{C}/(2\pi i X_*(G)) = G$.

14 How does inf char of π relate to $\rho|K^\times$?

Say F is the reals in this section, and K an algebraic closure of the reals. Say T is a torus over F, with induced lattice L, dual lattice $\widehat{L}$ and dual torus $\widehat{T}$ with induced action of Γ. Say ρ and π are related by local Langlands. Now ρ gives us a map $K^\times \to \widehat{T}$. Now if one chooses an isomorphism $K = \mathbf{C}$ then we get a map $\mathbf{C}^\times \to \widehat{T}$ which is a pair (λ, μ) of elements of $L \otimes \mathbf{C}$ whose difference is in L. The dictionary is that corresponding to λ, μ is the map $z \mapsto z^\lambda \overline{z}^\mu$, and $z^\lambda \overline{z}^\mu$ is supposed to be an element of $\mathrm{Hom}(\widehat{L}, \mathbf{C}^\times)$, and it is because if $\phi \in \widehat{L}$ then $(z^\lambda \overline{z}^\mu)(\phi) = z^{\phi(\lambda)} \overline{z}^{\phi(\mu)}$ and this makes sense because $\phi(\lambda - \mu) \in \mathbf{Z}$. Note that choosing the other isomorphism switches λ and μ. Furthermore, the fact that ρ has come from a representation of $W_{\mathbf{R}}$ means that $\mu = c.\lambda$ with $c \in \Gamma$ acting on L in the usual way.

Conclusion so far: ρ gives us a pair of elements of $L \otimes \mathbf{C}$ indexed by isomorphisms $K = \mathbf{C}$ and satisfying some properties (difference in L, one is c times the other).

Now let's consider π, a map $T(F) \to \mathbf{C}^\times$. We must remember here that L is the character group of T_K so we only get an isomorphism $L = X^*(T_{\mathbf{C}})$ once we choose an iso $K = \mathbf{C}$. Note that an element of L is a map $T(K) \to K^\times$, and choosing a different isomorphism is by definition changing that element of L by c, so different isomorphisms $K = \mathbf{C}$ produce different identifications of L with $X^*(T_{\mathbf{C}})$.

If h denotes the tangent space of $T(F)$ then the infinitesimal character of π is a map $h_{\mathbf{C}} \to \mathbf{C}$, and $h_{\mathbf{C}}$ is the tangent space of $T(\mathbf{C})$. If we choose $K = \mathbf{C}$ then $h_{\mathbf{C}}$ becomes the tangent space of $T(K)$, which is $\widehat{L} \otimes \mathbf{C}$, so the dual is $L \otimes \mathbf{C}$, and the inf char gives us such an element. The other choice of course just gives us c of the first choice. Although it's not immediately clear to me why the difference should be integral.... Are these supposed to be the same??

15 To do.

1) We have a definition of C-algebraic: that the inf char lands in the lattice plus δ. Check that Clozel's definition of algebraic for GL_n coincides with this one.

2) Work out what complex conjugation is going to! Split case is "clear", following Tate.

3) Toby said there was something else. I forgot.