Local Langlands for $\mathrm{GL}_2(\mathbf{R})$.

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A lot of this comes from the course I gave in 2004 on reps of real reductive groups. I added a little, but unfortunately not quite enough, about which g-K-modules were cohomological, on 29/04/2008.

Introduction

The local Langlands correspondence is a theorem for GL_n over the reals, I think (and for many if not all real or complex groups). Let's see how far I can get towards proving it for $\mathrm{GL}_2(\mathbf{R})$ by doing the naive approach: "write down all elements of both sides and match them up". I'll do the Galois theory in some huge generality and the representation theory in rather less generality.

1 Weil groups for the reals and the complexes.

The Weil group of a local field is supposed to relate to both the multiplicative group of the field, and also the absolute Galois group of the field. There is actually a purely axiomatic approach to the notion of the Weil group of a local (or indeed a global field). I won't follow it at all, I'll just give two concrete definitions. Let K be a finite extension of the real numbers.

(1) If $K \cong \mathbf{C}$ then define $W_K = K^\times$ (obvious topology).

(2) If $K \cong \mathbf{R}$ then define W_K to be the union of $\overline{K}^\times$ and $j\overline{K}^\times$ (j a formal symbol), with the rules that $j^2 = -1$ and $jcj^{-1} = \overline{c}$ (one checks easily that these rules are enough to tell you how to multiply any two elements of W_K together).

Note that in both cases, $\overline{K}^\times$ is a normal subgroup of W_K and the quotient group is canonically isomorphic to $\mathrm{Gal}(\overline{K}/K)$, giving an exact sequence

$$1 \to \overline{K}^\times \to W_K \to \mathrm{Gal}(\overline{K}/K) \to 1.$$

One key property that the Weil group of a local field K is supposed to have, at least if you know about the non-archimedean case, is that its abelianisation is supposed to come equipped with an isomorphism with $K^\times$. This is clear in the complex case. In the real case let's fix the isomorphism: one checks easily that the commutator subgroup of W_K is things of the form $c/\overline{c}$ with $c \in \mathbf{C}^\times$, that is, the unit circle in $\mathbf{C}^\times$. The quotient is hence naturally isomorphic to the union of $\mathbf{R}_{>0}$ and $j\mathbf{R}_{>0}$, and the isomorphism from $\mathbf{R}^\times$ to this sends -1 to j and $x > 0$ to $\sqrt{x}$. Alternatively, going the other way around, the isomorphism $W_{\mathbf{R}}^{ab} \to \mathbf{R}$ sends $z = x + iy \in \mathbf{C}^\times$ to $x^2 + y^2$. Note in particular that it doesn't depend on the choice of isomorphism $\overline{\mathbf{R}} = \mathbf{C}$. The "square root" (or square, depending on which way you're going) is for compatibility of this isomorphism under finite extensions of K; read Tate's article in Corvalis for more details.

Let's define a norm $||w||$ on W_K. If $K \cong \mathbf{C}$ then $||w|| = w\overline{w}$ (this doesn't depend on the choice of $K \cong \mathbf{C}$). If $K = \mathbf{R}$ then $||w||$ is $w\overline{w}$ for $w \in \mathbf{C}^\times$, and $||j|| = 1$. Note that the norm is a continuous group homomorphism $W_K^{ab} \to \mathbf{R}_{>0}$ which thus gives rise to (via our fixed isomorphisms) a continuous group homomorphism $K^\times \to \mathbf{R}_{>0}$, which turns out to be the norm coming from measure theory (that is, the function telling you how much multiplication expands a Haar measure on the additive group K) in both cases. Note also that in both cases, the kernel of the norm map is compact. Note finally that $W_{\overline{\mathbf{R}}} \subset W_{\mathbf{R}}$ and the norms agree.

By a *representation* of a Weil group we mean a continuous map into $\mathrm{GL}(V)$, with V a finite-dimensional complex vector space. We have lots of examples of 1-dimensional representations of Weil groups; if $s \in \mathbf{C}$ then consider $w \mapsto ||w||^s$ (note that if $r > 0$ is a positive real and $s \in \mathbf{C}$ then $r^s := \exp(s\log(r))$ makes sense). In fact there aren't too many more.

Lemma 1. *(a) The only continuous group homomorphisms $\mathbf{R} \to \mathbf{C}^\times$ are those of the form $x \mapsto \exp(sx)$, with distinct s giving distinct homomorphisms.*

(b) The only continuous group homomorphisms from the unit circle $S := \{z \in \mathbf{C} : |z| = 1\}$ to $\mathbf{C}^\times$ are of the form $z \mapsto z^n$ for some $n \in \mathbf{Z}$, with distinct n giving distinct homomorphisms.

(c) The only continuous group homomorphisms $\mathbf{R}_{>0} \to \mathbf{C}^\times$ are those of the form $x \mapsto x^s := \exp(s\log(x))$ for $s \in \mathbf{C}^\times$, with distinct s giving distinct homomorpsisms.

(d) The only continuous group homomorphisms $\mathbf{R}^\times \to \mathbf{C}^\times$ are of the form $x \mapsto x^{-N}||x||^s$ for $s \in \mathbf{C}$ and $N \in \{0, 1\}$, and distinct pairs (N, s) give distinct homomorphisms.

(e) The only continuous group homomorphisms $\mathbf{C}^\times \to \mathbf{C}^\times$ are of the form $z \mapsto z^{-N}||z||^s$ with $N \in \mathbf{Z}$ and $s \in \mathbf{C}$, and distinct pairs (N, s) give distinct homomorphisms.

Proof. Distinct data giving distinct homomorphisms is easy: just divide one representation by the other and the result is supposed to be 1.

(a) It suffices to prove that every group homomorphism $\mathbf{R} \to \mathbf{C}^\times$ agrees with $x \mapsto \exp(sx)$ on a small disc in $\mathbf{R}$. Now choose an open disc centre 1 radius 1 say; the pre-image of this contains an open neighbourhood of zero, say $(-\delta, \delta)$, and we can cut along the non-positive real axis and define a log on $\mathbf{C}^\times$ now, which is injective. We deduce the existence of a continuous "additive" (wherever this makes sense) map $(-\delta, \delta) \to \mathbf{C}$. Now say $\delta/2$ is sent to z; by continuity we see that the map is just multiplication by $2z/\delta$ on $[-\delta/2, \delta/2]$ and this is enough.

(b) Precomposing with the map $\mathbf{R} \to S$ given by $r \mapsto \exp(ir)$ we see that we need to classify the continuous group homomorphisms $\mathbf{R} \to \mathbf{C}^\times$ with 2π in the kernel; by (a) we just need to find all s such that $\exp(2\pi s) = 1$, that is, such that $2\pi s = 2\pi i n$ for some $n \in \mathbf{Z}$. We deduce that $s = in$ and that the representation is $\exp(ir) \mapsto \exp(inr)$ so we are done.

(c) Take logs and it follows from (a).

(d) Follows from (c).

(e) As a topological group, $\mathbf{C}^\times$ is the unit circle times $\mathbf{R}_{>0}$. For $\mathbf{R}_{>0}$ use (c); for the unit circle use (b). □

So we've now seen all the 1-dimensional representations of Weil groups. Tate is slightly more "canonical", not choosing an isomorphism $K = \mathbf{C}$ in the complex case—he says that if $K \cong \mathbf{C}$ then the 1-dimensional representations of W_K are all of the form $z \mapsto \sigma(z)^{-N}||z||^s$ with $\sigma : K \to \mathbf{C}$ an isomorphism (we need to use both isomorphisms to see all the reps though), $N \geq 0$, and $s \in \mathbf{C}$, and the only times distinct data gives the same isomorphism is when $N = 0$ in which case we don't mind which σ we choose. This normalisation is motivated by a study of L-functions and epsilon factors.

Having seen all the 1-dimensional representations of Weil groups, we move on to the higher dimensional case. By standard arguments, any continuous irreducible finite-dimensional representation of $W_{\mathbf{C}}$ has an eigenvector and is hence 1-dimensional, so we deduce that we have now seen all the irreducible n-dimensional representations of $W_{\mathbf{C}}$ and hence all the semisimple n-dimensional representations of $W_{\mathbf{C}}$.

For $W_{\mathbf{R}}$ there are some irreducible 2-dimensional representations. The point is that if ρ is an irreducible representation of $W_{\mathbf{R}}$ of dimension greater than 1 then the restriction of ρ to $W_{\mathbf{C}}$ must have an eigenvector, and if it's v then v and jv span an invariant subspace, so the dimension of ρ is 2, and ρ is induced from a character of $W_{\mathbf{C}}$. The easiest way of seeing what's going on is to note that any character of $W_{\mathbf{C}}$ is of the form $z \mapsto \sigma(z)^N||z||^s$ with $N \in \mathbf{Z}_{\geq 0}$ and if you induce this 1-dimensional representation then you get a 2-dimensional representation which is irreducible if $N > 0$, and reducible if $N = 0$. Conclusion: the irreducible representations of $W_{\mathbf{R}}$ are 1-dimensional of the form $W_{\mathbf{R}}^{\mathrm{ab}} = \mathbf{R}^\times \to \mathbf{C}^\times$ via $z \mapsto z^{-N}||z||^s$ with $N \in \{0, 1\}$ and $s \in \mathbf{C}$, and 2-dimensional induced from a character $z \mapsto \sigma(z)^{-N}||z||^s$ on $W_{\mathbf{C}}$, with $N \in \mathbf{Z}_{>0}$ and $s \in \mathbf{C}$.

Say that a representation of a Weil group W_K is "of Galois type" if it has finite image. This is iff it factors through $\mathrm{Gal}(\overline{K}/K)$. I'm not entirely sure how important these things are but that's the definition.

The irreducible representations of Weil groups above have L-functions and ϵ constants. You can just define these things via a list: for example in the complex case define $L(\sigma(z)^{-N}||z||^s) = 2(2\pi)^{-s}\Gamma(s)$; in the real abelian case define $L(x^{-N}||x||^s) = \pi^{-s/2}\Gamma(s/2)$ and so on. See Tate Corvalis for more information.

2 A crash course in functional analysis.

A *normed space* is a vector space (over the reals or complexes) with a real-valued norm $||.||$ on it satisfying $||x|| \geq 0$ with equality iff $x = 0$, the triangle inequality, and $||\lambda x|| = |\lambda|.||x||$ (if the base field is $\mathbf{C}$ then $|x+iy| = \sqrt{x^2+y^2}$). This induces a metric on the vector space. A Banach space is a normed space for which the metric is complete. One can complete a metric space and hence one can complete a normed space; the completion of a normed space is a Banach space and the map from the normed space into the Banach space is injective (Theorem 2.3-2 of Kreyszig). For example if $p \geq 1$ then the continuous real-valued functions on an interval $[a, b]$ with

$$||f|| = \left(\int_a^b |f(t)|^p \right)^{1/p}$$

is an incomplete normed space because it's easy to form a Cauchy sequence consisting of functions which converge to a step function. The completion is $L^p([a, b])$. Another construction of $L^p([a, b])$ is Lebesgue measurable functions f on $[a, b]$ such that the integral of $|f|^p$ on $[a, b]$ exists and is finite, modulo equivalence (quotient out by the subspace of things for which the integral is 0).

The dual of a normed space is the space of bounded (equivalently, continuous) linear functionals on the space. The dual space is always a Banach space (because the ground field is complete; $\mathrm{Hom}(X, Y)$ (continuous linear maps) is complete if Y is).

An *inner product space* is a vector space with an inner product $\langle \cdot, \cdot \rangle$ on it, which is a symmetric bilinear (resp. Hermitian sesquilinear) form (with values in the ground field, the reals or complexes), such that $\langle x, x \rangle \geq 0$ with equality iff $x = 0$. $L^2[a, b]$ is an example (both real and complex) with

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}dt$$

, as is ℓ^2 (sequences (a_n) with $\sum_n |a_n|^2 < \infty$). The inner product defines a norm via $||x|| = \sqrt{\langle x, x \rangle}$ and hence a metric; a Hilbert space is a complete inner product space. Every inner product space can be completed to a Hilbert space. The norm on an inner product space satisfies the *parallelogram equality* $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$ and one hence checks that there are plenty of normed spaces for which the norm doesn't come from an inner product.

If Y is a closed subspace of a Hilbert space V and $Z = Y^\perp$ then Z is also closed and $V = Y \oplus Z$. Via the abstract theory of orthonormal sets (and Zorn's Lemma) one sees that a Hilbert space is determined up to isomorphism by the cardinality of a "total orthonormal set" (Theorem 3.6-5 of Kreyszig). A separable Hilbert space over the complexes must be either finite-dimensional or ℓ^2. A Hilbert space is semi-linearly isomorphic to its dual: this is Riesz' Theorem (Theorem 3.8-1 of Kreyszig). More precisely, if V is a Hilbert space and f is a continuous functional on V then there is a unique $v \in V$ such that $f(x) = \langle v, x \rangle$ for all x.

If V and W are Hilbert spaces, and $T : V \to W$ is bounded, then it has an adjoint $T^* : W \to V$ which is also bounded, and characterised by $\langle Tv, w\rangle_W = \langle v, T^*w\rangle_V$. Note $(T^*)^* = T$. If $T : V \to V$ then T is *self-adjoint* if $T = T^*$, *unitary* if $TT^* = T^*T = 1$ and *normal* if $TT^* = T^*T$. Note that a continuous linear map $T : V \to V$ is unitary iff T is surjective and $\langle Tv, Tv\rangle = \langle v, v\rangle$ for all v. Note also that right-shifting on ℓ^2 satisfies $\langle Tv, Tw\rangle = \langle v, w\rangle$ for all v, w! See the bottom of p206 of Kreyszig.

3 Some infinite-dimensional representations of real and complex groups.

Let G be a topological group. A *representation* of G on a (possibly infinite-dimensional) complex Hilbert space V is a homomorphism ρ from G to $\mathrm{Aut}(V)$ (the group of linear maps $V \to V$ which are bijective, continuous, and have continuous inverses) such that the resulting map $G \times V \to V$ is continuous. We have the usual obvious definitions: an invariant subspace is the usual thing; V is *irreducible* if $V \neq 0$ and there are no *closed* invariant subspaces other than 0 or V. Say that a representation is *unitary* if for all $g \in G$ the map $\rho(g) : V \to V$ is unitary. The advantage of unitary representations is that the complement (wrt the inner product) of a closed invariant subspace is also closed and invariant.

One simple example: $\mathrm{SL}_2(\mathbf{R})$ acts on $L^2(\mathbf{R}^2)$ and $(\rho(g)f)(x) = f(g^{-1}x)$. This is easily checked to be unitary. Another natural representation is: if G is a group with a left Haar measure then choose a left Haar measure on G and let G act on $L^2(G)$ by $g.f(x) = f(g^{-1}x)$. This is also unitary.

Schur's Lemma works for unitary representations: a unitary Hilbert space representation is irreducible iff the only bounded endomorphisms of the Hilbert space commuting with the group action are the scalars. One way is easy but the other is tricky: if L is a non-scalar endomorphism of V commuting with the G-action we have to work a little to find an invariant subspace. Here's the idea: the adjoint L^* also commutes with the G-action because the rep is unitary; now $(L+L^*)/2$ and $(L-L^*)/2i$ also commute with the G-action, both are self-adjoint, and at least one is non-scalar. Now apply the spectral theorem (Chapter 9 of Kreyszig; this is some work to prove) to the non-scalar one and we get an idempotent E which is "a function of L" and hence also commutes with G.

Recall that if S is an open subset of $\mathbf{R}^n$ (for example a neighbourhood of the identity in G, if G is the real or complex points of a linear algebraic group) and f is a function from S to a topological real vector space V, then f is said to be differentiable at $s_0 \in S$ if there is a linear map $f'(s_0) : \mathbf{R}^n \to V$ such that $\lim_{s\to s_0} \frac{f(s)-f(s_0)-f'(s_0)(s-s_0)}{|s-s_0|} = 0$ where $|.|$ is any norm on $\mathbf{R}^n$. If f is differentiable at all $s \in S$ then the map $s \mapsto f'(s)$ is a map from S to $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, V) = V^n$ which is also a topological vector space, and we can ask whether this is differentiable. We say f is C^∞ if it's differentiable as many times as you like. If ρ is a representation of G (the points of a linear algebraic group) on a complex Hilbert space then we say $v \in V$ is a C^∞ vector for ρ if the map $G \to V$ defined by $g \mapsto \rho(g)v$ is C^∞. One easily checks that this is a complex subspace of V. It's not closed though, in fact in general just the opposite is true—if G is a Lie group then it's dense (Theorem 3.15 of Knapp, whose proof is basically elementary: one shows using a partition of unity argument that vectors can be approximated by vectors which are the solution to reasonable integrals, and that these integrals are C^∞). As a consequence one sees that finite-dimensional continuous reps of a Lie group are C^∞ automatically, which goes some way to explaining why Lemma 1 is true—continuous representations of reasonable groups are automatically infinitely differentiable, at least.

If $v \in C^\infty(V)$ then define $f : \mathfrak{g} \to V$ by $f(X) = \rho(\exp(X))v$; this is C^∞; define $\phi(X)v = f'(0)X$. That is,

$$\phi(X)v = \lim_{t\to 0} \rho(\exp(tX))v - vt.$$

If you now carefully unravel the definitions then you see that $\phi(X)$ sends $C^\infty(V)$ to itself, and that $\phi([X, Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X)$ (see Proposition 3.9 of Knapp). Hence the universal enveloping algebra acts on $C^\infty(V)$ as does it complexification. One checks that the centre of the universal enveloping algebra commutes with the G-action (corollary 3.12 of Knapp).

Now let G be a connected reductive affine algebraic group over $k = \mathbf{R}$ or $\mathbf{C}$. Then $G(k)$ has a maximal compact subgroup; call it K. Rather than prove this I'll just give examples. Note that K might not be the k-points of a closed subgroup of G, it's just an abstract group (it will be a real Lie group though).

$k = \mathbf{R}$: $\mathrm{GL}_n(\mathbf{R})$ contains $O(n)$; $\mathrm{SL}_n(\mathbf{R})$ contains $\mathrm{SO}(n)$; $\mathrm{Sp}_{2n}(\mathbf{R})$ contains a group isomorphic to $U(n)$; $k = \mathbf{C}$: $\mathrm{GL}_n(\mathbf{C})$ contains $U(n)$; $\mathrm{SO}_n(\mathbf{C})$ contains $\mathrm{SO}(n)$ (the matrices in $\mathrm{SO}_n(\mathbf{C})$ with real entries) and so on.

I am a bit muddled as to whether one can choose K arbitrarily in general; perhaps some of the theory only works if you choose the "correct" K; note that all the examples above are "correct" however!

If K is a compact topological group then the Peter–Weyl theorem (Theorem 1.12 of Knapp) implies that every irreducible unitary representation of K is finite-dimensional, and in fact more is true: every unitary Hilbert space representation of K has the property that one can write down a bunch of finite-dimensional invariant subspaces which are pairwise orthogonal and such that the closure of their direct sum is the whole representation. The proof is delicate but not too long; it uses an analysis of matrix coefficients (you can get away with the Stone-Weierstrass theorem if K is assumed to be a matrix group).

Now say G is linear connected reductive and K is a maximal compact subgroup of $G(k)$. If π is a representation of $G(k)$ on a complex Hilbert space V and if $v \in V$ then we say that v is *K-finite* if the $\mathbf{C}$-span of the set $\pi(K)v$ is finite-dimensional. If K acts by unitary operators then we can "see" this subspace. Firstly decompose the action of K on V into finite-dimensional invariant subspaces. If we choose an irreducible unitary representation τ of K then denote by V_τ the subspace of V spanned by subspaces of V isomorphic to τ. Now the K-finite vectors in V are just the algebraic direct sum of the V_τ as τ runs through the irreducible unitary representations of K in this case. We say that τ is a *type* for K if $V_\tau \neq 0$.

Theorem 2. *If G is linear connected reductive, K is a maximal compact subgroup of $G(k)$, and π is an irreducible unitary representation of $G(k)$ then each V_τ is finite-dimensional; in fact* $\dim(V_\tau) \leq \dim(\tau)^2$.

Proof. Knapp Theorem 8.1. □

Definition. A representation of a linear connected reductive group $G(k)$ on a Hilbert space V is *admissible* if $\pi(K)$ acts as unitary operators and if each irreducible unitary representation τ of K occurs only finitely often in $\pi|K$.

So the previous theorem says that irreducible unitary representations are admissible. There are non-unitary irreducible admissible representations too, in general: general induced representations seem to have this property, as do most finite-dimensional representations! In fact admissible is a neat trick which catches both infinite-dimensional unitary and finite-dimensional representations, isn't it. I suspect that historically people studied unitary representations and I suspect that it was Langlands who might have introduced the notation of admissibility?

Theorem 3. *Let V be an admissible repn of linear connected reductive G. Then the K-finite vectors in V are all C^∞, and of course they might not be G-stable, but they are $\mathfrak{g}$-stable.*

Proof. Proposition 8.5 of Knapp. One shows analogous to Knapp Theorem 3.15 (used above) that the C^∞ K-finite vectors are dense, and then uses admissibility to finish. □

Note that the reason the K-finite vectors are not G-stable is that if v is K-finite then gv is gKg^{-1}-finite and gKg^{-1} might not be commensurable with K. This is one big big difference between the archimedean and non-archimedean cases.

We say that two admissible representations of G are *infinitesimally equivalent* if the associated representations of $\mathfrak{g}$ on the K-finite vectors are algebraically equivalent. Infinitesimally equivalent does not imply isomorphic, it's weaker, so I'm told. But infinitesimally equivalent irreducible unitary representations are indeed equivalent: the essence of this argument is rather easy (Corollary 9.2 of Knapp).

Lemma 4. *If π is an admissible representation of G on V and V_0 is the K-finite vectors, then V_0 is dense in V and, even better, there's a natural bijection between the closed G-invariant subspaces U of V and the algebraic $\mathfrak{g}$-invariant subspaces U_0 of V_0, the dictionary being $U_0 = U \cap V_0$ and $U = \overline{U_0}$. In particular, V is irreducible iff V_0 is.*

Proof. Corollary 8.10/8.11 of Knapp and the remarks at the top of p212. □

If V is admissible with no non-trivial closed invariant subspaces then we say that π is irreducible admissible. Langlands classified irreducible admissible representations of G up to infinitesimal equivalence! I don't know what his theorem is though. One nice fact (the Casselman Subrepresentation Theorem, Theorem 8.37 in Knapp) is that every irreducible admissible representation of a linear connected reductive group is infinitesimally equivalent with a subrepresentation of some (possibly non-unitary) principal series representation (a principal series representation is, vaguely speaking, a representation induced from a twist of a unitary representation of some parabolic subgroup; see p168 of Knapp. Principal series representations induced from irreducible unitary representations are admissible, by Proposition 8.4 of Knapp).

If π is irreducible admissible then the centre of the complexified universal enveloping algebra acts as a scalar on the K-finite vectors of π (Corollary 8.14 of Knapp).

We've seen that from an admissible representation, the K-finite vectors admit a representation of $\mathfrak{g}$ and K. Here's a formal definition.

Let K a be a compact subgroup of $G(k)$, G any affine algebraic group. A $(\mathfrak{g}, K)$-*module* is a complex vector space V equipped with an action of $\mathfrak{g}_{\mathbf{C}}$ and K such that the K-representation is a (possibly infinite) algebraic direct sum of finite-dimensional representations of K (that's equivalent to every vector being K-finite), the actions are compatible in the sense that if $X \in \mathfrak{g}$ is in the Lie algebra of K then for all $v \in V$ we have Xv (the $\mathfrak{g}$-action) is the derivative with respect to t of $\exp(tX)v$ at $t = 0$, and if $k \in K$ and $X \in \mathfrak{g}_{\mathbf{C}}$ and $v \in V$ then $kXv = ((\mathrm{ad}\,k)(X))(kv)$ (here k acts on G and hence on $\mathfrak{g}$ by conjugation).

A $(\mathfrak{g}, K)$-module is *admissible* if for all representations τ of K the number of times τ shows up in V is finite. A submodule is the obvious thing. Clearly an admissible representation of G gives an admissible $(\mathfrak{g}, K)$-module by taking K-finite vectors. Note that according to Wallach Corvallis, Corollary 4.19, any admissible irreducible $(\mathfrak{g}, K)$-module is the K-finite vectors in an admissible irreducible (Hilbert space) representation of G. Wallach deduces this from the Casselman subrepresentation theorem.

A key example: let V be an admissible Hilbert space representation of $G(k)$ and let V_0 denote the K-finite vectors in V. Now vectors in V_0 are C^∞ and hence $\mathfrak{g}$ (and also $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$) acts on V_0, as does K, and furthermore the actions of $\mathfrak{g}_{\mathbf{C}}$ and K are compatible, in the following sense: firstly, if $X \in \mathfrak{g}$ satisfies $\exp(X) \in K$ then the two definitions of the action of X (one via thinking it of as in $\mathfrak{g}$, the other via differentiating the K-action) are the same, and secondly the action of kX is the same as $(kXk^{-1})k$, where kXk^{-1} has a meaning in the Lie algebra, as k acts on G and hence on $\mathfrak{g}$ by conjugation. So V_0 gets the structure of an admissible $(\mathfrak{g}, K)$-module.

Later on I will classify all irreducible $(\mathfrak{gl}_2, O(2))$-modules; they will all turn out to be admissible. Richard Taylor once told me that irreducible implies admissible for all reductive groups G.

Here are some other definitions, while I'm here. Let V be an irreducible admissible representation of $G(k)$ with G connected linear reductive; let V_0 be the K-finite vectors. If we choose $v, w \in V$ then we get a function $G \to \mathbf{C}$ defined by $g \mapsto (gv, w)$ and this is called a *matrix coefficient.* If $v, w \in V_0$ then it's called a *K-finite matrix coefficient.* If all K-finite matrix coefficients are in $L^2(G)$ then we say that V is *discrete series.* In this case V is infinitesimally equivalent to the action of G on an irreducible closed subspace of $L^2(G)$ (Theorem 8.51 of Knapp; see also Proposition 9.6 of Knapp). Note that if V is irreducible and unitary then it's enough to check that one non-zero K-finite matrix coefficient is in $L^2(G)$.

If the K-finite matrix coefficients are in $L^{2+\epsilon}(G)$ for all $\epsilon > 0$ then we say that V is *irreducible tempered.* Such a representation is infinitesimally equivalent with a unitary representation. See Theorem 8.53 of Knapp for other facts about irreducible tempered representations. Some induced representations are tempered, and others aren't.

4 The classification of $(\mathfrak{g}, K)$-modules in the GL_2 case.

I want to write down all the irreducible admissible $(\mathfrak{g}, K)$-modules in the case $G = \mathrm{GL}_2$ over $\mathbf{R}$, so $\mathfrak{g} = \mathfrak{gl}_2(\mathbf{R})$ and $K = O(2)$. Note that K isn't connected. Let's define $K_0 = \mathrm{SO}(2)$. Say V is an irreducible $(\mathfrak{g}, K_0)$-module. Because K_0 is isomorphic to the circle, via $e^{i\theta} = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$, we see that $V = \oplus V_n$ where $V_n = \{v \in V : e^{i\theta}.v = e^{ni\theta}v\}$. Now K_0 acts on $\mathfrak{g}_{\mathbf{C}}$ by conjugation but unfortunately the eigenspaces are a bit messy, so we have to grit our teeth. Set $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$. Then $\gamma^{-1}\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)\gamma = \left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$ which does act nicely on our favourite basis $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and so on. So let's choose a basis $z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $e = \gamma\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\gamma^{-1}$ and $f = \gamma\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\gamma^{-1}$ and $h = \gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\gamma^{-1}$ for $\mathfrak{g}$. Recall $[h, e] = 2e$ and $[h, f] = -2f$ and $[e, f] = h$ and everything commutes with z. We also now have that if $k = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$ then $kek^{-1} = e^{2i\theta}e$, $kfk^{-1} = e^{-2i\theta}f$ and $khk^{-1} = h$. Note also that $\exp(i\theta h)$ is just $\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \in K$ and using the definition of a $(\mathfrak{g}, K)$-module gives us that ih should act via the derivative at zero with respect to θ of this, which on V_n is in, so h acts as multiplication by n on V_n.

Let H, E, F, Z denote the corresponding elements of the universal enveloping algebra. Set $\Omega = (H-1)^2 + 4EF$. Now in $U(\mathfrak{g})$ we have $EF - FE = H$ so this is also $(H+1)^2 + 4FE$. One checks that Ω is actually in the centre of $U(\mathfrak{g})$; this is easy, we know $U(\mathfrak{g})$ is generated by Z, E, F, H and e.g. $\Omega E = H^2E - 2HE + E + 4EFE = H(HE - 2E) + E + 4EFE$ and $HE - EH = 2E$ so this is $HEH + E + 4EFE = (2E + EH)H + E + 4EFE = E(H^2 + 2H + 1 + 4FE) = E\Omega$ and so on. In fact the centre of $U(\mathfrak{g}_{\mathbf{C}})$ is just $\mathbf{C}[\Omega, Z]$—this will not be important for the classification, but we'll use it later. Because V is irreducible, Ω must act by a scalar, call it ω, and Z must act by a scalar, call it z (abuse of notation but this is OK).

Now if $v \in V_n$ then $Ev \in V_{n+2}$ because $kEv = (kEk^{-1})kv$, and similarly $Fv \in V_{n-2}$. Also $EF = \frac{1}{4}(\Omega - (H-1)^2)$ and hence $EFv = \frac{1}{4}(\omega - (n-1)^2)v$

if $v \in V_n$ and similarly $FEv = \frac{1}{4}(\omega - (n+1)^2)v$. In particular we see that if $\omega - (n-1)^2 \neq 0$ and $0 \neq v \in V_n$ then $0 \neq Fv \in V_{n-2}$ either.

Now *if* there exists $0 \neq v \in V_n$ such that $Fv = 0$ then $\oplus_{r \geq 0} E^r \mathbf{C} v$ is stable under E, F, H, Z and K_0, so it's V and note that this forces $\omega = (n-1)^2$. Similarly if there exists $0 \neq v \in V_n$ with $Ev = 0$ then $V = \oplus_{r \geq 0} F^r \mathbf{C} v$ and $\omega = (n+1)^2$. Conversely, if $\omega = (n-1)^2$ and $V_n \neq 0$ then $FV_n = 0$, and if $\omega = (n+1)^2$ and $V_n \neq 0$ then $EV_n = 0$, and in either case we now know V.

We now see that if $\omega \neq m^2$ for any $m \in \mathbf{Z}$ then either $V = \oplus_{n \in 2\mathbf{Z}} V_n$ or $V = \oplus_{n \in 2\mathbf{Z}+1} V_n$ and in both cases the dimension of all the V_n in the sum is 1 (the point is that if $0 \neq v \in V_n$ then $E^r v$ and $F^r v$ can't vanish, and the space generated by these vectors is stable). On the other hand if $\omega = m^2$ for some $m \in \mathbf{Z}_{\geq 0}$ (WLOG) then V is one of the following four things: $V = \oplus_{n \equiv m(2)} V_n$ or $V = \oplus_{n=-\infty, n \not\equiv m(2)}^{-m-1} V_n$ or $V = \oplus_{n=1-m, n \not\equiv m(2)}^{m-1} V_n$ (with $m \geq 1$) or $V = \oplus_{n=m+1, n \not\equiv m(2)}^{\infty} V_n$ and all the V_n mentioned in the sum are 1-dimensional.

One deduces from this that a $(\mathfrak{g}, K_0)$-module is determined by ω, z, and a type for the module (that is, an irreducible representation τ of K_0 which occurs as a subrepresentation of the $(\mathfrak{g}, K_0)$-module), and furthermore for each choice of ω, z, and type, there's a unique irreducible $(\mathfrak{g}, K_0)$-module with these parameters. We have also proved that irreducible implies admissible and that for fixed ω and z (that is, for fixed infinitesimal character) there are only finitely many $(\mathfrak{g}, K_0)$-modules. This fact is generally true: see Corollary 10.37 of Knapp.

Now say V is an irreducible $(\mathfrak{gl}_2, \mathrm{O}_2)$-module, and set $c = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Now V is either irreducible or reducible as a $(\mathfrak{gl}_2, \mathrm{SO}_2)$-module. If it's irreducible (case A) then $cV_n = V_{-n}$ as $cHc^{-1} = -H$, so V can't be $\oplus_{n=-\infty, n \not\equiv m(2)}^{-m-1} V_n$ or $\oplus_{n=m+1, n \not\equiv m(2)}^{\infty} V_n$ (although it could be the finite-dimensional option). In all of the remaining cases precisely one of V_0 and V_1 is non-zero, and one can easily check that in all cases there are only finitely many possibilities for the action of c, up to isomorphism. Let's do this: either (case A1) $\omega = m^2$ with $m \in \mathbf{Z}_{\geq 0}$ and either $V = \oplus_{n \in \mathbf{Z}, n \equiv m \bmod 2} \mathbf{C} v_n$ or $m > 0$ and $V = \oplus_{n=1-m}^{m-1} \mathbf{C} v_n$ and there are exactly two ways in which c can act. Or (case A2) $\omega \notin \mathbf{Z}^2$. Then $V = \oplus_{n \in \mathbf{Z}, n \equiv t \bmod 2} \mathbf{C} v_n$ and V is determined by z, ω, $t \bmod 2$, and a choice of a c-action (two choices again).

The alternative (case B) is that V is reducible as an $(\mathfrak{gl}_2, \mathrm{SO}_2)$-module and if $0 \subset W \subset V$ is a submodule then $V = W \oplus cW$ with $cW \not\equiv W$ and checking cases shows us that we are forced to have $\omega = m^2$ with $m \in \mathbf{Z}_{\geq 0}$ and $V = \oplus_{n \leq -m-1, n \geq m+1, n \equiv m+1 \bmod 2} \mathbf{C} v_n$.

5 Jacquet–Langlands' description of these modules.

If B is a Borel in a connected linear algebraic group G, and V is a (say, unitary Hilbert space) representation of $B(\mathbf{R})$, then it would be nice to know how to induce it up to a representation of G but there are apparently some minor technical measure-theory problems with this induction and I won't explain how to do it properly (you look at continuous functions from G to V which transform in a certain way and then complete in the L^2 norm, so it seems). On the other hand, induction for $(\mathfrak{g}, K)$-modules seems to be much easier—I imagine this is the same as taking the usual induction and then taking the K-finite vectors. I haven't found a reference for induction on $(\mathfrak{g}, K)$-modules but here's something concrete which must be an example, which I got from Jacquet–Langlands.

Recall from Lemma 1 that the only continuous group homomorphisms $\mathbf{R}^\times \to \mathbf{C}^\times$ are of the form $\mu(x) = |x|^s(x/|x|)^N$ with $s \in \mathbf{C}$ and $N \in \{0,1\}$. Let μ_1, μ_2 be two such characters. Consider the pair as a character of the upper triangular matrices in $\mathrm{GL}_2(\mathbf{R})$. Now induce up: set

$$\begin{aligned} B(\mu_1, \mu_2) =&\{f : \mathrm{GL}_2(\mathbf{R}) \to \mathbf{C} : \\ &f\left(\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right) g\right) = \mu_1(a)\mu_2(b)|a/b|^{1/2} f(g) \text{ and } f \text{ is } \mathrm{SO}_2(\mathbf{R})\text{-finite}\}. \end{aligned}$$

The finiteness statement is that if f satisfies the equation above then for all $k \in \mathrm{SO}_2(\mathbf{R})$ we see that the function $f_k : \mathrm{GL}_2(\mathbf{R}) \to \mathbf{C}$ defined by $f_k(g) = f(gk)$ also satisfies the equation above, but we want the vector space generated by the f_k as k runs through K to be finite-dimensional.

Define s and N by $\mu_1\mu_2^{-1} = |t|^s(t/|t|)^N$. Now one checks that if $n \in \mathbf{Z}$ with $n \equiv N$ mod 2 then the function ϕ_n defined by $\phi_n(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)) = \mu_1(a)\mu_2(b)|a/b|^{1/2}e^{in\theta}$ is in $B(\mu_1, \mu_2)$ and in fact these functions form a basis of $B(\mu_1, \mu_2)$. Note that $B(\mu_1, \mu_2)$ is an admissible $(\mathfrak{gl}_2, \mathrm{O}_2)$-module. Moreover by our classification of irreducible ones (which involves working out the dictionary), or just by looking at what Jacquet-Langlands do on p164ff do, we can read off when these things are irreducible. If $s - N$ isn't an odd integer, or if it is but $s = 0$, then $B(\mu_1, \mu_2)$ is irreducible. If however $s - N$ is an odd integer and $s \neq 0$ then there are two cases. Either $s > 0$ in which case one checks that $\sigma(\mu_1, \mu_2) := \oplus_{n>=s+1, n<=-s-1}\mathbf{C}\phi_n$ is an irreducible sub and the quotient $\pi(\mu_1, \mu_2)$ is finite-dimensional and irreducible. If on the other hand $s < 0$ then $1 + s \leq n \leq -1 - s$ gives a finite-dimensional irreducible submodule, call it $\pi(\mu_1, \mu_2)$, and the quotient $\sigma(\mu_1, \mu_2)$ is irreducible. One now checks without too much trouble (it's a long long check though) that every irreducible admissible $(\mathfrak{gl}_2(\mathbf{R}), \mathrm{O}_2)$-module is either a π or a σ, and that the only isomorphisms between them are $\pi(\mu, \nu) = \pi(\nu, \mu)$ and $\sigma(\mu, \nu) = \sigma(\nu, \mu) = \sigma(\mu\eta, \nu\eta) = \sigma(\nu\eta, \mu\eta)$ where $\eta(t) = t/|t|$. The dictionary is that if $\mu_i(t) = |t|^{s_i}(t/|t|)^{N_i}$ then $z = s_1 + s_2$ and $\omega = (s_1 - s_2)^2$ on $B(\mu_1, \mu_2)$, and it's easy to unravel the rest now (the N_i are a bit of a pain).

Note that there is something "simple" going on here whose analogue is much more complicated in the p-adic case. In the p-adic case we could induce up characters of the Borel, and we got principal series and special representations, but there were also strange supercuspidal representations which didn't arise as a subquotient of principal series. In the case of $\mathrm{GL}_2(\mathbf{R})$ this doesn't happen.

6 L-functions of irreducible admissible representations.

There are two conventions here, unfortunately. Jacquet and Langlands associate L-functions to representations; Tate associates numbers to representations, and then twisting the representations by $|.|^s$ gives rise to functions. Fortunately Tate's numbers are just Jacuqet-Langlands' functions evaluated at zero, so let's go with Jacquet-Langlands. Note that Jacquet and Langlands attempt to associate an L-function to the σs and πs in the $\mathrm{GL}_2(\mathbf{R})$ case via an "explicit" construction re-interpreting the representations as certain spaces of functions on $\mathrm{GL}_2(\mathbf{R})$ a la Whittaker model, but I've not tried to understand this and will instead just write down the answer.

6.1 $\mathrm{GL}_1/\mathbf{R}$

The (Langlands) L-function of $x \mapsto |x|^r.(x/|x|)^m$ (with $m \in \{0,1\}$) is $\pi^{-(s+r+m)/2}\Gamma((s+r+m)/2)$. Note that if m has the right parity but isn't in $\{0,1\}$ then this formula is *wrong*.

6.2 $\mathrm{GL}_1/\mathbf{C}$

The Langlands L-function of $z \mapsto z^m(z\overline{z})^r$ with $m \in \mathbf{Z}_{\geq 0}$ is $2(2\pi)^{-(s+r+m)}\Gamma(s+r+m)$. Note that if $m < 0$ is an integer then again this formula is *wrong*. In fact what I have said above isn't enough to determine all L-functions: we also need to know that the L-function of $z \mapsto \overline{z}^n(z\overline{z})^r$ with $r \in \mathbf{C}$ and $n \in \mathbf{Z}_{\geq 0}$ is $2(2\pi)^{-(s+r+n)}\Gamma(s+r+n)$. Note that if $m = n = 0$ then the characters agree but so do the L-functions.

On p194 of Jacquet-Langlands they also explain the epsilon factors.

6.3 $\mathrm{GL}_2/\mathbf{R}$

The L-function of $\pi(\mu_1, \mu_2)$ is just the product $L(\mu_1, s)L(\mu_2, s)$.

Now if $\sigma(\mu_1, \mu_2)$ exists then (after possibly switching μ_1 and μ_2, and possibly multiplying them both by η, the function $x \mapsto (x/|x|)$) we can write $\mu_1(t) = |t|^{r+(m/2)}$ with $r \in \mathbf{C}$ and $m \in \mathbf{Z}_{>0}$ and $\mu_2(t) = |t|^{r-(m/2)}\eta(t)^{m-1}$. Define $\omega : \mathbf{C}^\times \to \mathbf{C}^\times$ by $\omega(z) = (z\overline{z})^{r-(m/2)}z^m$ (we could also use $(z\overline{z})^{r-(m/2)}\overline{z}^m$) and define $L(\sigma(\mu_1, \mu_2) = L(\omega, s)$ (I just told you what this was in the previous subsection: it's $2(2\pi)^{-(s+r+(m/2))}\Gamma(s+r+(m/2))$.

7 The Local Langlands correspondence for $\mathrm{GL}_2/\mathbf{R}$.

I've just mentioned, or implied, what the correspondence is. The reducible representation $\mu_1 \oplus \mu_2$ becomes associated with $\pi(\mu_1, \mu_2)$, and if $m \in \mathbf{Z}_{>0}$ then the irreducible 2-d Weil representation induced from the map $\mathbf{C}^\times \to \mathbf{C}^\times$

defined by $z \mapsto z^m(z\overline{z})^s$ is attached to $\sigma(\mu_1,\mu_2)$ with $\mu_1(t) = |t|^(s+m)$ and $\mu_2(t) = |t|^s.(t/|t|)^(m-1)$.

The Harish-Chandra isomorphism at infinity is a map from the centre of $U(\mathfrak{g}^{\mathbf{C}})$ to $U(\mathfrak{h}^{\mathbf{C}})^W$ (with W the Weyl group and the superscript $\mathbf{C}$ meaning "tensor with $\mathbf{C}$") and it sends Ω to H^2 and sends Z to Z. Moreover it seems to me that the lattice of characters in $\mathfrak{h}\otimes\mathbf{C}$ (that is, the ones coming from the torus) is the $\mathbf{C}$-linear maps which send Z and H to integers of the same parity. Note also that half the sum of the positive roots is not in this lattice: it sends Z to zero and H to 1. So $\pi(\mu_1,\mu_2)$ or $\sigma(\mu_1,\mu_2)$ are algebraic iff $\mu_i(t) = |t|^{s_i}(t/|t|)^{N_i}$ with $s_1, s_2 \in \mathbf{Z}$.

8 The Local Langlands theorem for $\mathrm{GL}_2/\mathbf{R}$.

Recall that all of the irreducible $(\mathfrak{gl}_2, \mathrm{O}_2)$-modules we wrote down do show up as the K-finite vectors in a Hilbert space representation of $\mathrm{GL}_2(\mathbf{R})$ (see Wallach Corvallis 4.19); we deduce

Theorem 5. *There's a natural bijection between the 2-dimensional representations of $W_{\mathbf{R}}$ and infinitesimal equivalence classes of admissible representations of $\mathrm{GL}_2(\mathbf{R})$.*

The representations associated to weight k modular forms are called σ_k, and these are the ones with $V = \oplus_{n=k,n\equiv k(2)}^{\infty}(V_n \oplus V_{-n})$, with $\omega = (k-1)^2$ and $z = 2-k$. If $k > 1$ then this is called a discrete series representation, and if $k = 1$ it's called a limit of discrete series. The Jacquet–Langlands name for σ_k, $k > 1$, will be $\sigma(\mu_1,\mu_2)$ with $\mu_1(x) = |x|^{1/2}$ and $\mu_2(x) = |x|^{\frac{3-2k}{2}}(x/|x|)^k = x^{2-k}|x|^{-1/2}$, if $k > 1$, where we note that this corresponds to an *irreducible* representation of the Weil group; note that $(\mu_1/\mu_2)(x) = x^{k-1}(x/|x|)$ so we induce the representation $z \mapsto z^{k-1}$ of $W_{\mathbf{C}}$ to $W_{\mathbf{R}}$. The J–L name for σ_1 is $\pi(\mu_1,\mu_2)$ with $\mu_1(x) = |x|^{1/2}$ and $\mu_2(x) = |x|^{1/2}(x/|x|)$.

Waffle: To get ones hands on what's going on in the general case, one has to understand induction properly. Basically one should stick to inducing twists of unitary representations, so it seems. And here there's a trick: there is a unitary discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ corresponding to the σ's. One can write it down explicltly: given an integer $s \geq 1$ one defines D_s^+ to be the representation of $\mathrm{SL}_2(\mathbf{R})$ acting on the analytic f on the upper half plane and satisfying $||f|| = \int\int |f(z)|^2 y^{s-1}dxdy < \infty$, with the action being $(gf)(z) = (bz+d)^{-(s+1)}f((az+c)/(bz+d))$ and then one induces this up to $\mathrm{SL}_2^{\pm}(\mathbf{R})$ and gets a representation of $\mathrm{GL}_2(\mathbf{R})$ in what I presume is the usual way. Now one thinks about the πs as coming from induced representations from the Borel, but the σs as coming from twists of these discrete series. I don't really understand why. It seems to me that any irreducible admissible representation of GL_n should be infinitesimally equivalent to something induced from a Borel but the way people explain it is to induce from a slightly bigger group. Somehow it fits best into Langlands' framework in this setting. Langlands proved that inducing up twists of tempered unitary representations from parabolics gave all irreducible admissible representations of an arbitrary reductive group G up to infinitesimal equivalence. This is known as Langlands' classification of irreducible admissible representations. To deal with the tempered representations though, one runs into the theory of L-packets about which I know nothing.

9 $(\mathfrak{g}, K)$-cohomology in the GL_2 case.

I'll quote freely from Borel-Wallach (the 2000 edition). Let's deal with the centre first. Let $\mathfrak{g}$ be the 1-dimensional (abelian) Lie algebra over the complexes, and let V be a 1-dimensional complex vector space with an action of $\mathfrak{g}$. Choose a basis $1 \in \mathfrak{g}$.

Lemma 6. *If $1 \in \mathfrak{g}$ acts as multiplication by $s \in \mathbf{C}$ on V then $z \in \mathfrak{g}$ acts as zs. The Lie algebra cohomology groups $H^i(\mathfrak{g}; V)$ of V are either zero for all $i \geq 0$, or $s = 0$ in which case $H^0(\mathfrak{g}; V)$ and $H^1(\mathfrak{g}; V)$ are both 1-dimensional and all the others are zero-dimensional.*

Proof. By definition the Lie algebra cohomology is the cohomology of a complex $C^*(\mathfrak{g}; V) = \mathrm{Hom}_{\mathbf{C}}(\Lambda^*(\mathfrak{g}), V)$ which is zero in degrees other than 0 and 1, and is 1-dimensional in degrees zero and 1. The result will follow if we check that d is an isomorphism iff $s \neq 0$, and this follows straight from the definition of d in equation (2) of Chapter 1 of Borel-Wallach, which shows that $d = s$. □

Much more interesting is the following. Let $\mathfrak{g}$ now denote the complexified Lie algebra of $\mathrm{SL}_2(\mathbf{R})$, let K denote $\mathrm{SO}_2(\mathbf{R})$ and let $\mathfrak{k}$ denote its complexified Lie algebra. Choose a basis X, Y, H of $\mathfrak{g}$ such that $\mathbf{C}H = \mathfrak{k}$ and $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$ (I am thinking $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $Y = X^t$ and $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). Let V denote an admissible representation of $\mathrm{SL}_2(\mathbf{R})$ over the complexes.

Lemma 7. *The $(\mathfrak{g}, \mathfrak{k})$-cohomology of V is the cohomology of the following complex of finite-dimensional complex vector spaces:*

$$V^{H=0} \to V^{H=2} \oplus V^{H=-2} \to V^{H=0}$$

where the first map sends v to (Xv, Yv) and the second sends (a, b) to $Ya - Xb$.

Proof. Again one simply applies the definition of the complex defining $(\mathfrak{g}, \mathfrak{k})$-cohomology; it is by definition the cohomology of this complex. See section 1.2 of Borel-Wallach. □

Corollary 8. *If f is a modular form of weight 3 or more, or weight 1, then the associated $(\mathfrak{g}, K)$-module has no cohomology.*

Proof. The associated $(\mathfrak{sl}_2, \mathfrak{k})$-module has $V^{H=0} = V^{H=2} = V^{H=-2} = 0$ because $V^{H=n}$ is only non-zero for $|n| \geq k$ and $n \equiv k \bmod 2$. Now the Künneth formula (section 1.3 of BW) shows that there will be no cohomology as a $(\mathfrak{gl}_2, \mathfrak{k})$-module either. But $(\mathfrak{gl}_2, K)$-cohomology is the K-invariants of $(\mathfrak{gl}_2, \mathfrak{k})$-cohomology so this also vanishes. □