Level raising

Kevin Buzzard

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1 Introduction

What's at the heart of level raising, when working with 1-dimensional Shimura varieties? I ask this because I'm trying to understand Rajaei's thesis. Firstly I'll write down the abstract algebra lemma which shows that level-raising will be a consequence of a certain module being Eisenstein. Then some waffle about Rajaei's thesis but no conclusions. Note that the abstract algebra lemma appears to be only the right thing when the underlying Shimura variety is a curve—in the 0-dimensional case the combinatorics are a bit different and in particular we don't get an alternating pairing on anything as far as I know. However, these notes are mainly my attempt to understand what Rajaei must be doing so I'm not worried by this.

2 The abstract algebra lemma

Let $\mathcal{O}$ be a complete DVR with field of fractions K, and let L and M be finite free $\mathcal{O}$-modules. Say L is equipped with an alternating perfect $\mathcal{O}$-linear pairing—that is, an $\mathcal{O}$-bilinear map $\langle,\rangle_L : L \times L \to \mathcal{O}$ such that $\langle \alpha, \beta \rangle_L = -\langle \beta, \alpha \rangle_L$ and such that the induced $\mathcal{O}$-linear map $L \to \mathrm{Hom}_{\mathcal{O}}(L, \mathcal{O})$ defined by

$$\lambda \mapsto (\alpha \mapsto \langle \alpha, \lambda \rangle_L)$$

is an isomorphism of finite free $\mathcal{O}$-modules. Say M is also equipped with an alternating perfect $\mathcal{O}$-linear pairing $\langle,\rangle_M$.

Say $i : L \to M$ is an injective $\mathcal{O}$-module homomorphism. Note however that we make *no* assumption about how i relates to the pairings on L and M. Let $i^* : M \to L$ denote the dual homomorphism—more precisely, this means that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\cong} & \mathrm{Hom}(M,\mathcal{O}) \\ \downarrow{\scriptstyle i^*} & & \downarrow \\ L & \xrightarrow{\cong} & \mathrm{Hom}(L,\mathcal{O}) \end{array}$$

commutes, where the right hand map is the one induced by i and the horizontal isomorphisms are those given by the pairings, namely $\mu \mapsto (\beta \mapsto \langle \beta, \mu \rangle_M)$ and $\lambda \mapsto (\alpha \mapsto \langle \alpha, \lambda \rangle_L)$. A more concise definition of i^* is that it's the unique map satisfying $\langle \lambda, i^*\mu \rangle_L = \langle i\lambda, \mu \rangle_M$ for all $\lambda \in L, \mu \in M$. Note that even though i is injective, i^* may not be surjective—for example if $L = M$ and i is multiplication by $\lambda \in \mathcal{O}$ then so is i^*. Note also that because both $\langle , \rangle_L$ and $\langle , \rangle_M$ are alternating we have $\langle i^*\mu, \lambda \rangle_L = -\langle \lambda, i^*\mu \rangle_M = -\langle i\lambda, \mu \rangle_M = \langle \mu, i\lambda \rangle_L$.

For an $\mathcal{O}$-module N, let N_K denote $N \otimes_{\mathcal{O}} K$. The final assumption that we make on the situation is that the composite map $i^*i : L \to L$ is injective, and hence induces an isomorphism $i^*i : L_K \to L_K$ of finite-dimensional K-vector spaces. Let $j : L_K \to L_K$ denote the inverse $(i^*i)^{-1}$ of i^*i on L_K. The pairing on L extends to an alternating pairing (also called $\langle , \rangle_L$ on the K-vector space L_K and we see that i^*i is self-adjoint with respect to this pairing as $\langle i^*i\lambda_1, \lambda_2 \rangle_L = \langle i\lambda_1, i\lambda_2 \rangle_M = \langle \lambda_1, i^*i\lambda_2 \rangle_L$. Hence j is also self-adjoint with respect to this pairing.

The pairing on M gives rise to a bilinear form $\langle , \rangle_M$ on the K-vector space M_K. Let M_K^{old} denote $i(L_K)$ and let M_K^{new} denote $\{m \in M_K : \langle m, \mu \rangle_M = 0 \text{ for all } \mu \in M_K^{\mathrm{old}}\}$. Note that $M_K = M_K^{\mathrm{old}} \oplus M_K^{\mathrm{new}}$, because by a dimension count it suffices to prove $M_K^{\mathrm{old}} \cap M_K^{\mathrm{new}} = 0$, and if $i(\lambda)$ were in the kernel for $\lambda \in L_K$ then λ would be in the kernel of i^*i. In particular, the pairing induced on M_K^{old} by $\langle , \rangle_M$ is non-degenerate (because anything in the kernel would be in M_K^{new}).

Here are four $\mathcal{O}$-lattices in M_K^{old}:

$$\Lambda_0 := ijL.$$

$$\Lambda_1 := iji^*M.$$

$$\Lambda_2 := M \cap M_K^{\mathrm{old}}.$$

$$\Lambda_3 := iL.$$

I claim that $\Lambda_3 \subseteq \Lambda_2 \subseteq \Lambda_1 \subseteq \Lambda_0$. The first and last of these inclusions follow immediately from the fact that $i : L \to M$ and $i^* : M \to L$. The middle inclusion follows from the fact that $iji^* : M_K \to M_K$ is a projection (i.e., it squares to itself, an easy calculation) with image M_K^{old}.

These lattices have a nice duality property, which is purely formal (although I seem to be making a bit of a meal of it below):

Lemma 1. *The perfect pairing $\langle,\rangle_M$ on M_K^{old} induces isomorphisms $\Lambda_i \to \mathrm{Hom}_{\mathcal{O}}(\Lambda_{3-i}, \mathcal{O})$.*

Proof. The general element of M_K^{old} is of the form $i(\lambda)$ for $\lambda \in L_K$, and the map $M_K \to K$ induced by $i(\lambda)$ is the map $\mu \mapsto \langle \mu, i(\lambda)\rangle_M$. In particular the lattice corresponding to $\mathrm{Hom}(\Lambda_0, \mathcal{O})$ is the $i\lambda$ in M_K such that for all $\lambda' \in L$ we have $\langle ij\lambda', i\lambda\rangle_M \in \mathcal{O}$. Now

$$\langle ij\lambda', i\lambda\rangle_M = \langle i^*ij\lambda', \lambda\rangle_L = \langle \lambda', \lambda\rangle_L$$

and for $\lambda \in L_K$ we have $\langle \lambda', \lambda\rangle_L \in \mathcal{O}$ for all $\lambda' \in L$ iff $\lambda \in L$ (as $\langle,\rangle_L$ is a perfect duality on L). Hence Λ_3 is identified with $\mathrm{Hom}_{\mathcal{O}}(\Lambda_0, \mathcal{O})$ and by duality Λ_0 is identified with $\mathrm{Hom}_{\mathcal{O}}(\Lambda_3, \mathcal{O})$.

Similarly the lattice corresponding to $\mathrm{Hom}(\Lambda_1, \mathcal{O})$ is the $\mu \in M_K^{\mathrm{old}}$ such that $\langle iji^*\mu', \mu\rangle_M \in \mathcal{O}$ for all $\mu' \in M$, and because j is adjoint with respect to $\langle,\rangle_L$ we may move each operator to the right hand side and see $\langle iji^*\mu', \mu\rangle_M = \langle \mu', iji^*\mu\rangle_M$. Furthermore iji^* is the identity on M_K^{old} and hence $\langle \mu', iji^*\mu\rangle_M = \langle \mu', \mu\rangle_M$. We deduce that $\mathrm{Hom}(\Lambda_1, \mathcal{O})$ is identified with the μ in M_K^{old} such that $\langle \mu', \mu\rangle_M \in O$ for all $\mu' \in M$, which is $M_K^{\mathrm{old}} \cap M = \Lambda_2$, and by duality $\mathrm{Hom}(\Lambda_2, \mathcal{O})$ is identified with Λ_1. □

The basic phenomenon of level-raising is the following: there will be some Hecke algebra $\mathbf{T}$ acting on both L and M, and i will be a $\mathbf{T}$-module homomorphism (note that $\mathbf{T}$ will not have a Hecke operator at the prime we are adding, or if it does then its action on M will be defined rather delicately). Maximal ideals of $\mathbf{T}$ in the support of Λ_1/Λ_2 will be "new". Maximal ideals in the support of Λ_0/Λ_3 are easily understood, because $\Lambda_0/\Lambda_3 = jL/L = L/i^*iL$ and typically i and i^* are easily computed. Vaguely speaking, maximal ideals in the support of Λ_0/Λ_3 are those satisfying a congruence condition (typically, the determinant of i^*i will be in $\mathbf{T}$ and the condition will just be that this determinant is in the maximal ideal).

To raise the level, one has to prove that all "interesting" maximal ideals satisfying the congruence condition are in the support of Λ_1/Λ_2. Equivalently, what one needs to do is to check that all the maximal ideals in the support of Λ_0/Λ_1 and Λ_2/Λ_3 are "uninteresting". The word usually used for "uninteresting" is "Eisenstein", which in applications will mean that the mod ℓ Galois representations associated to the maximal ideals are reducible. Proving that this is the case is frequently the deepest part of the argument (perhaps even the only deep part of the argument). Note however that these two modules above are dual to one another:

Lemma 2. *There is a perfect pairing*

$$\Lambda_0/\Lambda_1 \times \Lambda_2/\Lambda_3 \to K/\mathcal{O}.$$

Proof. The pairing $\langle,\rangle_M$ on M_K^{old} induces a map $\Lambda_0 \times \Lambda_2 \to K/\mathcal{O}$ and by the previous lemma, the left and right kernels of this map are exactly Λ_1 and Λ_3. □

In any given case, unravelling the way that this pairing behaves with respect to the $\mathbf{T}$-action gives enough information to conclude that one only need to check that one of these modules is Eisenstein—and the one usually chosen is Λ_2/Λ_3.

3 The classical case.

In Ribet's original work on this problem, he considered cases where L and M were the first etale cohomology groups of certain modular curves, with constant ℓ-adic coefficients (although Ribet worked consistently with Jacobians). Ribet's observation was that in these cases, one could understand Λ_2/Λ_3 via the snake lemma applied to the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "i"] & L_K \arrow[r] \arrow[d, "i"] & L_K/L \arrow[r] \arrow[d, "i"] & 0 \\ 0 \arrow[r] & M \arrow[r] & M_K \arrow[r] & M_K/M \arrow[r] & 0 \end{tikzcd} .$$

Because the middle vertical arrow has no torsion in its kernel or cokernel, we see that Λ_2/Λ_3 is the torsion in M/iL, which is precisely the kernel of $i : L_K/L \to M_K/M$. Unravelling the classical definition of the Jacobian shows that if $L = H^1(X_1, \mathbf{Z}_l)^2$ and $M = H^1(X_2, \mathbf{Z}_l)$, with the X_i smooth projective curves, and i is induced by two finite maps $X_2 \to X_1$, and if J_i is the Jacobian of X_i, then L_K/L is exactly the torsion in J_1^2, M_K/M is the torsion in J_2, and the kernel of $L_K/L \to M_K/M$ is the kernel of the induced map $J_1^2 \to J_2$. The heart of the proof then is to understand the kernel of this morphism of Jacobians.

In Ribet's case, $X_1 = X(U)$ and $X_2 = X(U_0(p))$ where U was a level structure prime to p and $U_0(p)$ was $U \cap \Gamma_0(p)$. Ribet showed that if the level structure was $\Gamma_1(N)$ or $\Gamma(N)$ then the map on Jacobians was injective, and hence $\Lambda_2 = \Lambda_3$ (and so $\Lambda_0 = \Lambda_1$). If however the level structure was $\Gamma_0(N)$ then he proved that the quotients were non-trivial but managed to identify them. In his ICM paper he computed their order and concluded level-raising results for maximal ideals of residue characteristic not dividing this order, but later on (in his Seminaire de Theorie des Nombres Paris 1987–88 paper) he observed that the kernels were Eisenstein and hence level-raising always worked for irreducible representations. Note however that Ribet's argument controlling the kernels was *not* purely formal. Ribet reduces everything to showing $\Lambda_2 = \Lambda_3$ in the case $U = \Gamma(N)$, and then deduces this equality from very delicate facts about finite index subgroups of $\mathrm{PSL}_2(\mathbf{Z}[1/p])$, facts which basically turn out to be equivalent to the congruence subgroup property for this latter group.

4 Taylor's argument in the totally definite quat alg case.

There are some differences here. Firstly, Taylor's pairing is not alternating, although when it's not alternating it's symmetric, and this doesn't change any of the arguments, so that difference is merely cosmetic. Secondly, his pairing isn't perfect either! However Taylor is not trying to prove congruences mod p, he's trying to prove them mod p^n for arbitrarily large n in some sense, but can vary the prime he's trying to add, and so (because he's in fixed weight) the failure of the pairing to be perfect is controllable (the cokernels of all the maps $L \to \mathrm{Hom}(L, \mathcal{O})$ and those for the Ms he considers (he considers more than one because he varies the prime he's adding) are all bounded by a constant depending only on the weight). His argument is basically the same; the heart of the argument is something analogous to proving $\Lambda_2 = \Lambda_3$: it's a check that Λ_3/Λ_2 is annihilated by some non-zero element of the base ring which is independent of the prime he's adding.

5 The Hilbert and the unitary case

Rajaei seems to prove Ribet's result in the Hilbert case. Nothing appears to be written in the unitary case. Note that Rajaei allows ℓ to divide the level (which isn't so surprising as the geometry is going on at p), he allows arbitrary weights, in particular weights $k > \ell$, which is perhaps more surprising because for example Jordan and Livné didn't, and he allows ℓ to be ramified in F, which perhaps isn't surprising but may be something I won't be able to deal with when removing the final prime from the level.

Rajaei wants to add a prime $\mathfrak{p}$ to the level in the Hilbert case, with ℓ not equal to the prime below $\mathfrak{p}$. Let U denote a level structure prime to $\mathfrak{p}$ (let's be vague and work in either the quaternion or $U(1,1)$ case) and let $U_0(\mathfrak{p})$ denote the level structure that you get when you add $\mathfrak{p}$ to U. Assume U is sufficiently small for there to be no representability issues. Let X_U and $X_{U_0(\mathfrak{p})}$ denote the two modular curves associated to the two level structures, and consider these curves at the minute as being defined over a number field E (totally real or CM, depending). Let $\mathcal{F}$ denote the ℓ-adic sheaf on X_U or $X_{U_0(\mathfrak{p})}$ corresponding to the weight k we're interested in (Toby probably is only interested in the trivial sheaf; Ali works in much more generality). We start with an ℓ-adic representation occurring in $H^1(X_U \times \overline{E}, \mathcal{F})$ and what we want to understand is when there is a $\mathfrak{p}$-new representation congruent mod ℓ to this representation occurring in $H^1(X_{U_0(\mathfrak{p})} \times \overline{E}, \mathcal{F})$.

Let L denote the torsion-free quotient of $H^1(X_U \times \overline{E}, \mathcal{F})^2$ and let M denote the torsion-free quotient of $H^1(X_{U_0(\mathfrak{p})} \times \overline{E}, \mathcal{F})$. My understanding is that in all the situations we are interested in, $L \otimes \overline{\mathbf{Q}}_l$ will be the direct sum of the Galois representations associated to the level U weight k automorphic forms—the new ones will occur once, and the old ones will perhaps show up more than once, in fact one can easily predict how many times they show up via the theory of the conductor. I am not sure how correct this is in the unitary case, although I am pretty confident about things in the Hilbert case. If corresponds to a general weight k then I strongly suspect that $H^1(X_U \times \overline{E}, \mathcal{F})$ can have torsion—consider it as group cohomology and then the point is that $\mathcal{F}/l\mathcal{F}$ could have fixed vectors which don't lift to $\mathcal{F}$ and this is exactly the phenomenon that gives torsion in H^1 by a long exact sequence argument. In particular if the weights are bigger than ℓ then this could occur. On the other hand, anything coming from H^0 will be Eisenstein so probably shouldn't worry us. What we are worried about is the non-formal analysis of Λ_3/Λ_2.

We have two degeneracy maps $Y_{U_0(\mathfrak{p})} \to Y_U$ and this induces a map $L \to M$. This map will be injective in all cases we consider, for example by tensoring with $\overline{\mathbf{Q}}_l$ and then looking at what's going on on the level of automorphic forms? Again I am a bit worried about the unitary case because I don't really understand if there is a theory of a conductor in all cases. Perhaps I should be assuming in the unitary case that all the primes in the level split in the quadratic extension.

My goal now is to see how Rajaei analysis Λ_3/Λ_2 in his case, and see if it works in the unitary case too.