An analogue of the BGG resolution for overconvergent p-adic automorphic forms

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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To Lavinia.

Abstract

We study maps between spaces of overconvergent p-adic automorphic forms for groups which are compact modulo centre at all infinite places. This is motivated by pioneering work of Coleman. By defining maps between spaces of overconvergent p-adic modular forms he was able to prove a classicality criterion which was fundamental in studying p-adic families of modular forms.

Spaces of overconvergent p-adic automorphic forms for groups which are compact modulo centre at all infinite places are defined by applying an exact functor to certain representations called locally analytical principal series for the Iwahori subgroup. We therefore study maps between these representations.

We prove a relation between these representations and certain well-known infinite dimensional Lie algebra representations called Verma modules. In particular, we show that all maps between these representations come from maps between Verma modules in an explicit way. Using standard theorems about maps between Verma modules, it follows that the space of maps we are interested in generically has dimension zero. Its dimension is non-zero only for certain prescribed one-dimensional representations, when its dimension is one. We can now define maps, called theta maps, between spaces of overconvergent p-adic automorphic forms of different weights but the same tame level for a group which is compact modulo centre at all infinite places.

We also prove an analogue of the BGG resolution for locally analytic principal series for the Iwahori subgroup. This allows us to use theta maps to give a classicality criterion for overconvergent p-adic automorphic forms for groups which are compact modulo centre at all infinite places.

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Chapter 1

Introduction

Locally analytic representation theory is the study of a certain class of representations of L-analytic groups over K, where L is a finite extension of $\mathbb{Q}_p$ and K is a spherically complete extension of L. It was systematically developed by Schneider and Teitelbaum in papers such as [17], [19], [20] and [21]. It plays an important role in the p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$.

It also has applications to overconvergent p-adic automorphic forms. For connected reductive groups which are compact modulo centre at infinity, spaces of overconvergent p-adic automorphic forms are defined by Loeffler in [14] in terms of functions from a certain set to a locally analytic principal series representation for an Iwahori subgroup.

Let G be the group of L-points of a connected reductive linear quasisplit algebraic group defined over L, B and $\overline{B}$ opposite Borel subgroups in G, G_1 an open subgroup of G admitting an Iwahori factorisation, such as an Iwahori subgroup, and $\overline{B}_1 = \overline{B} \cap G_1$. We study maps between locally analytic principal series $\mathrm{Ind}_{\overline{B}_1}^{G_1}(\mu)$ for G_1, where Ind denotes locally analytic induction over K, which we assume is complete with respect to a discrete valuation. Our approach is to exploit an isomorphism between $\mathrm{Ind}_{\overline{B}}^{G}(\mu)(N)$, the subspace of functions in $\mathrm{Ind}_{\overline{B}}^{G}(\mu)$ with support in $\overline{B}N$, where N is the unipotent radical of B, and $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)$, the space of locally analytic functions $N \to K_\mu$ with compact support, where K_μ is a one-dimensional K-vector space with an action of B coming from $\mu \in X(\mathbf{T})$. Using the dense subspace $\mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_\mu) \subseteq \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)$, which we call the space of locally polynomial functions, we prove the following theorem.

Theorem 30. *We have an exact sequence of M-representations*

$$0 \to V \otimes_K \text{sm-Ind}_{B_1}^{G_1}(\mathbf{1}) \to \text{Ind}_{B_1}^{G_1}(\lambda) \to \bigoplus_{w \in W^{(1)}} \text{Ind}_{B_1}^{G_1}(w \cdot \lambda) \to \cdots \to \bigoplus_{w \in W^{(i)}} \text{Ind}_{B_1}^{G_1}(w \cdot \lambda) \to \cdots \to \text{Ind}_{B_1}^{G_1}(w_0 \cdot \lambda) \to 0$$

coming from the BGG resolution for V^*.

Here M is a particular submonoid of G containing the G_1, V is the irreducible finite-dimensional algebraic representation of G with highest weight λ, sm-Ind is smooth induction, $\mathbf{1}$ is the trivial character, $W^{(i)}$ denotes the elements of the Weyl group of length i, w_0 is the longest element of the Weyl group and $w \cdot \lambda = w(\lambda + \rho) - \rho$ where ρ is half the sum of the positive roots.

By taking G_1 to be an Iwahori subgroup, we can use Theorem 30 to construct the analogous exact sequence for locally analytic principal series $\text{Ind}_B^G(\mu)$ for G, which has been established by quite different methods in [16]. Another consequence of Theorem 30 is the following exact sequence between spaces $M(e, K_\mu)$ of overconvergent p-adic automorphic forms of weight μ for a group compact modulo centre at infinity whose group of L-points is G.

Theorem 39. *If $\lambda \in X(\mathbf{T})$ is dominant and arithmetical then we have a Hecke-equivariant exact sequence*

$$0 \to M(e, K_\lambda)^{cl} \to M(e, K_\lambda) \to \bigoplus_{w \in W^{(1)}} M(e, K_{w \cdot \lambda}) \to \cdots \to \bigoplus_{w \in W^{(i)}} M(e, K_{w \cdot \lambda}) \to \cdots \to M(e, K_{w_0 \cdot \lambda}) \to 0.$$

Here $M(e, K_\lambda)^{\text{cl}}$ denotes the so-called classical subspace. After the first inclusion, the maps in this exact sequence are constructed from maps of the form $\theta^{\text{aut}}_{\alpha, w \cdot \lambda} : M(e, K_{w \cdot \lambda}) \to M(e, K_{s_\alpha w \cdot \lambda})$, where $w \in W$ and $\alpha \in \Phi^+$ satisfy $l(s_\alpha w) = l(w) + 1$. These are the analogue of the maps θ^{k-1} from [6] between the spaces of overconvergent p-adic modular forms of weight $2 - k$ and k. In [6], Coleman used θ^{k-1} to prove a sufficient condition for classicality in terms of small slope. Using Theorem 39 we can establish a necessary and sufficient condition for belonging to the classical subspace.

1.1 Structure

In Chapter 2 we define certain subspaces of locally analytic principal series in which we will be interested. In Chapter 3 we establish results about certain categories $\mathcal{O}$ and $\overline{\mathcal{O}}$ of representations of a split semisimple Lie algebra, including the exactness of a certain duality functor between them. In Chapter 4 we use maps between Verma modules to construct maps between particular subspaces of locally analytic principal series. We use the BGG resolution in Chapter 5 to construct a sequence of $\mathcal{U}(\mathfrak{g})$-modules involving these subspaces. We then show that the first three terms of this sequence are exact in Chapter 6. In Chapter 7 we prove Theorem 30, from which we deduce the exactness of the original sequence, and prove that the first three terms of the exact sequence in Theorem 30 remain exact when we restrict to analytic principal series. We prove the analogue of Theorem 30 involving locally analytic principal series for G in Chapter 8.

Finally, we give some applications of our results to overconvergent p-adic automorphic forms for groups compact modulo centre at infinity. In Chapter 9 we briefly sketch the definition of overconvergent p-adic automorphic forms given by Chenevier in [5] and prove a three-term exact sequence involving certain spaces of overconvergent automorphic forms. This material is contained in [5], citing an earlier version of our work, but is included here for completeness. In Chapter 10 we briefly outline the definition of overconvergent p-adic automorphic forms given by Loeffler in [14] and prove Theorem 39.

1.2 Notation

Fix a prime p. Let L be a finite extension of $\mathbb{Q}_p$ and let K be an extension of L which is complete with respect to a discrete valuation. Lemma 1.6 in [18] implies that K is spherically complete.

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over L which is quasi-split over L and split over K. Choose a Borel subgroup $\mathbf{B}$ which is defined over L. Write

$\mathbf{N}$ for its unipotent radical (which is defined over L). Choose a maximal L-split torus in $\mathbf{B}$ and let $\mathbf{T}$ be its centraliser in $\mathbf{G}$. Then $\mathbf{T}$ is a Levi factor in $\mathbf{B}$ and a maximal torus in $\mathbf{G}$ which is defined over L. It is not necessarily split over L, but by assumption it splits over K. Let $\overline{\mathbf{B}}$ denote the opposite Borel to $\mathbf{B}$ containing $\mathbf{T}$ and $\overline{\mathbf{N}}$ its unipotent radical.

We write G for $\mathbf{G}(L)$. We use bold letters to denote algebraic subgroups of $\mathbf{G}$. For any algebraic subgroup $\mathbf{J}$ of $\mathbf{G}$ defined over L we write J to denote $\mathbf{J}(L)$ and the lower case gothic letter $\mathfrak{j}$ to represent the corresponding Lie subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. The sole exception is that we will denote the Lie algebra of T by $\mathfrak{h}$, which is the standard notation in Lie algebra representation theory. Given a Lie algebra $\mathfrak{a}$ we write $\mathcal{U}(\mathfrak{a})$ for the universal enveloping algebra of $\mathfrak{a}$. Representations of $\mathfrak{a}$ are equivalent to $\mathcal{U}(\mathfrak{a})$-modules, and we use the two terms interchangeably. We write $S : \mathcal{U}(\mathfrak{a}) \to \mathcal{U}(\mathfrak{a})$ for the principal anti-automorphism of $\mathcal{U}(\mathfrak{a})$, given on monomials by $X_1 \cdots X_n \mapsto (-1)^n X_n \cdots X_1$. This is the unique algebra anti-automorphism of $\mathcal{U}(\mathfrak{a})$ extending $\mathfrak{a} \to \mathfrak{a}$, $X \mapsto -X$.

If $J \subset G$ has an action on a $\mathcal{U}(\mathfrak{g})$-module which is differentiable such that the two actions of $\mathfrak{j}$ agree then we call the $\mathcal{U}(\mathfrak{g})$-module a $(\mathfrak{g}, J)$-module. Maps between $(\mathfrak{g}, J)$-modules which are $\mathcal{U}(\mathfrak{g})$-equivariant and J-equivariant are called $(\mathfrak{g}, J)$-equivariant.

Let Φ denote the set of all roots of $\mathbf{G}$ with respect to $\mathbf{T}$, $\Phi^+ \subseteq \Phi$ the subset of positive roots determined by our choice of $\mathbf{B}$ and $\Delta \subseteq \Phi^+$ the corresponding set of simple roots. Let $r = |\Phi^+|$. Set $\Phi^- = \{-\alpha : \alpha \in \Phi^+\}$. For each $\alpha \in \Phi^+$ let $H_\alpha \in \mathfrak{h}$ denote its coroot and fix a non-zero element $E_\alpha \in \mathfrak{g}_\alpha$. This determines a unique $F_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[E_\alpha, F_\alpha] = H_\alpha$.

Let $X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$, with the group law written additively. As $\mathbf{T}$ is split over K, all $\mu \in X(\mathbf{T})$ are defined over K. Let $\mathfrak{h}^*$ be the space of Lie algebra homomorphisms from $\mathfrak{h}$ to K. Any $\mu \in X(\mathbf{T})$ gives an element in $\mathfrak{h}^*$ by evaluating at K-points, restricting to T and then differentiating at the identity. We denote this element again by μ. We are mainly interested in elements of $\mathfrak{h}^*$ coming from $X(\mathbf{T})$.

Let W denote the Weyl group of $\mathbf{G}$ and $\mathbf{T}$, $W^{(i)}$ the subset of elements of length i under the Bruhat ordering given by our choice of positive roots and w_0 the longest element of W. Let ρ be half the sum of the positive roots. There is a natural action of W on $X(\mathbf{T})$, and we define the affine action of $w \in W$ on $\mu \in X(\mathbf{T})$ by $w \cdot \mu = w(\mu + \rho) - \rho$. We define the affine action of W on $\mathfrak{h}^*$ similarly.

If χ is a locally analytic character $T \to GL_1(K)$ then let K_χ denote the one-dimensional representation of B over K given by $B \to B/N \cong T \xrightarrow{\chi} GL_1(K)$ and let A_χ denote the one-dimensional representation of $\overline{B}$ over K given by $\overline{B} \to \overline{B}/\overline{N} \cong T \xrightarrow{\chi} GL_1(K)$. We are mostly interested in K_μ and A_μ for $\mu \in X(\mathbf{T})$.

Suppose X is a paracompact locally L-analytic manifold and U a K-vector space. We write $\mathcal{C}^{\mathrm{la}}(X, U)$ for the space of locally L-analytic functions from X to U, and $\mathcal{C}^{\mathrm{sm}}(X, U)$ for the subspace of all smooth (i.e. locally constant) functions. The subspaces of compactly supported functions are denoted $\mathcal{C}^{\mathrm{la}}_{\mathrm{c}}(X, U)$ and $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(X, U)$ respectively. If Ω is an open and closed subset of X then we write $\mathbf{1}_\Omega \in \mathcal{C}^{\mathrm{sm}}(X, K)$ for the indicator function of Ω. For any $f \in \mathcal{C}^{\mathrm{la}}(X, U)$ we write $f_{|\Omega}$ for $f\mathbf{1}_\Omega$. If Y is an open subset of X, $\mathbb{Y}$ is a rigid analytic space defined over L and $\varphi : Y \to \mathbb{Y}(L)$ is a locally analytic isomorphism which is compatible with all the charts of X then we write $\mathcal{C}^{\mathrm{an}}(Y, K)$ for the subspace of $\mathcal{C}^{\mathrm{la}}(Y, K)$ consisting of functions $f : Y \to K$ such that $f \circ \varphi^{-1}$ comes from a holomorphic function on $\mathbb{Y}(L)$. We say $f \in \mathcal{C}^{\mathrm{la}}(X, K)$ is analytic on Y if $f|_Y \in \mathcal{C}^{\mathrm{an}}(Y, K)$.

Chapter 2

Subspaces of $\mathrm{Ind}_{\overline{B}}^{G}(\mu)$

Now we recall some definitions and propositions from [8], Emerton's forthcoming paper on the relation of his Jacquet module functor to parabolic induction, which contains a longer exposition of all the material in this chapter. For simplicity we often give definitions only in the cases we need them, rather than the more general versions found in [8]. All representations will be vector spaces over K, even if this is not explicitly mentioned.

Definition 1. *Let U be a barrelled, Hausdorff, locally convex K-vector space with an action of a locally L-analytic group J by continuous K-linear automorphisms. We say U is a* ***locally analytic representation*** *of J if for every $u \in U$ the orbit map $J \to U, j \mapsto ju$ is in $\mathcal{C}^{la}(J, U)$.*

If U is a locally analytic representation of H then we can differentiate the action of J to get an action of $\mathfrak{j}$, or equivalently of its enveloping algebra $\mathcal{U}(\mathfrak{j})$, as explained in Remark 2.5 in [11].

If χ is a locally analytic character $T \to GL_1(K)$ then we define the locally analytic parabolic induction of A_χ from $\overline{B}$ to G to be

$$\mathrm{Ind}_{\overline{B}}^{G}(\chi) = \{f \in \mathcal{C}^{\mathrm{la}}(G, K) : f(\overline{n}tg) = \chi(t)f(g) \text{ for all } \overline{n} \in \overline{N}, t \in T, g \in G\}$$

with the right regular action of G: $g'f(g) = f(gg')$. This is a locally analytic representation of G, as explained in Proposition 2.1.1 of [8]. We are interested in the case $\chi = \mu \in X(\mathbf{T})$.

The support of any $f \in \mathrm{Ind}_{\overline{B}}^{G}(\mu)$ is an open and closed subset of G which is invariant under multiplication on the left by $\overline{B}$. Its image in $\overline{B}\backslash G$ is therefore open and compact. We refer to this as the support of f, $\mathrm{Supp}\, f$. If Ω is any open subset of $\overline{B}\backslash G$ we let $\mathrm{Ind}_{\overline{B}}^{G}(\mu)(\Omega)$ denote the subspace of elements whose support is contained in Ω.

Since $N \cap \overline{B} = \{e\}$, the natural map $N \to \overline{B}\backslash G$ given by $n \mapsto \overline{B}n$ is an open immersion. We use this map to regard N as an open subset of $\overline{B}\backslash G$. By Lemma 2.3.6 of [8], this open immersion induces a topological isomorphism

$$\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu) \xrightarrow{\sim} \mathrm{Ind}_{\overline{B}}^{G}(\mu)(N). \tag{2.1}$$

We extend the right translation action of N on $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)$ to a locally analytic action of B by letting $t \in T$ act on $f \in \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)$ as follows:

$$tf(n) = \mu(t) f(t^{-1}nt).$$

On the other hand the action of B on $\mathrm{Ind}_{\overline{B}}^{G}(\mu)$ preserves $\mathrm{Ind}_{\overline{B}}^{G}(\mu)(N)$ as $\overline{B}NB = \overline{B}N$, so we have an action of B on $\mathrm{Ind}_{\overline{B}}^{G}(\mu)(N)$. These actions make (2.1) B-equivariant.

As $\mathrm{Ind}_{\overline{B}}^{G}(\mu)$ is a locally analytic representation of G it also has an action of $\mathfrak{g}$. If $X \in \mathfrak{g}$ and $f \in \mathrm{Ind}_{\overline{B}}^{G}(\mu)$ is 0 on some open neighbourhood of $g \in G$ then Xf is also 0 on this neighbourhood. Hence the action of $\mathfrak{g}$ preserves $\mathrm{Ind}_{\overline{B}}^{G}(\mu)(N)$, whence we get an action of $\mathfrak{g}$ on $\mathrm{Ind}_{\overline{B}}^{G}(\mu)(N)$. Restricting it to $\mathfrak{b}$ gives the same action as differentiating the B-action, so $\mathrm{Ind}_{\overline{B}}^{G}(\mu)(N)$ is a $(\mathfrak{g}, B)$-module. We use (2.1) to transfer this action of $\mathfrak{g}$ to $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)$.

We now identify various subspaces of $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)$, which by (2.1) correspond to subspaces of $\mathrm{Ind}_{\overline{B}}^{G}(\mu)(N)$.

Define $\mathcal{C}^{\mathrm{pol}}(N, K)$, the ring of algebraic K-valued functions on N, to be the set of all functions $N \to K$ which come from global sections of the structure sheaf of $\mathbf{N}$ over K. We give $\mathcal{C}^{\mathrm{pol}}(N, K)$ its finest locally convex topology, so the natural injection into $\mathcal{C}^{\mathrm{la}}(N, K)$ is continuous. We let N act by the right regular representation. We extend this to an action of B by $tf(n) = f(t^{-1}nt)$. This makes $\mathcal{C}^{\mathrm{pol}}(N, K)$ an algebraic representation, in the sense that we may write it as a union of an increasing series of finite dimensional B-invariant subspaces, on each of which B acts through an algebraic representation of $\mathbf{B}$. Each of these representations is a fortiori a locally analytic representation of B, so we can differentiate them to get actions of $\mathfrak{b}$. These all agree, so we get an action of $\mathfrak{b}$ on $\mathcal{C}^{\mathrm{pol}}(N, K)$. In fact because we have given $\mathcal{C}^{\mathrm{pol}}(N, K)$ its finest locally convex topology the action of B makes it a locally analytic representation, as explained after Lemma 2.5.3 in [8], which gives us another way of constructing this action of $\mathfrak{b}$.

We define the space $\mathcal{C}^{\mathrm{pol}}(N, K_\mu)$ of polynomial functions on N with coefficients in K_μ to be $\mathcal{C}^{\mathrm{pol}}(N, K) \otimes_K K_\mu$, equipped with the inductive, or equivalently projective, tensor product topology (cf. §17 of [18]). This has an action of B by letting it act on both factors. Since both of these actions are locally analytic the action on $\mathcal{C}^{\mathrm{pol}}(N, K_\mu)$ is locally analytic, so we have an action of $\mathfrak{b}$. We now explain how to extend this to an action of $\mathfrak{g}$.

We write $\mathfrak{n}^k$ for $\{E_1 \cdots E_k : E_i \in \mathfrak{n} \text{ for all } 1 \leq i \leq k\} \subseteq \mathcal{U}(\mathfrak{n})$. For any $\mathcal{U}(\mathfrak{n})$-module M we define $M^{\mathfrak{n}^\infty}$ to be the subspace of all $x \in M$ such that $\mathfrak{n}^k x = \{0\}$ for some positive integer k. From the discussion following Lemma 2.5.3 in [8] the map

$$\mathcal{C}^{\mathrm{pol}}(N, K_\mu) \xrightarrow{\sim} \mathrm{Hom}_K(\mathcal{U}(\mathfrak{n}), A_\mu)^{\mathfrak{n}^\infty} \qquad\qquad f \longmapsto (u \mapsto (uf)(e))$$

is an isomorphism of $\mathcal{U}(\mathfrak{n})$-modules, where the action of $\mathfrak{n}$ on $\mathrm{Hom}_K(\mathcal{U}(\mathfrak{n}), A_\mu)$ is given by $X\phi(u) = \phi(uX)$ for all $X \in \mathfrak{n}$, $\phi \in \mathrm{Hom}_K(\mathcal{U}(\mathfrak{n}), K_\mu)$ and $u \in \mathcal{U}(\mathfrak{n})$. By the Poincaré-Birkhoff-Witt theorem there is an isomorphism of $\mathcal{U}(\mathfrak{n})$-modules $\mathrm{Hom}_K(\mathcal{U}(\mathfrak{n}), A_\mu) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}(\bar{\mathfrak{b}})}(\mathcal{U}(\mathfrak{g}), A_\mu)$ where $\mathcal{U}(\mathfrak{g})$ is a $\mathcal{U}(\bar{\mathfrak{b}})$-module by the multiplication on the left and has an action of $\mathfrak{n}$ by multiplication on the right. Combining these we get an isomorphism of $\mathcal{U}(\mathfrak{n})$-modules

$$\mathcal{C}^{\mathrm{pol}}(N, K_\mu) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}(\bar{\mathfrak{b}})}(\mathcal{U}(\mathfrak{g}), A_\mu)^{\mathfrak{n}^\infty}.$$

We give $\mathrm{Hom}_{\mathcal{U}(\bar{\mathfrak{b}})}(\mathcal{U}(\mathfrak{g}), A_\mu)^{\mathfrak{n}^\infty}$ an action of $\mathfrak{g}$ by $X\phi(u) = \phi(uX)$ for all $X \in \mathfrak{g}$, $\phi \in \mathrm{Hom}_{\mathcal{U}(\bar{\mathfrak{b}})}(\mathcal{U}(\mathfrak{g}), A_\mu)^{\mathfrak{n}^\infty}$ and $u \in \mathcal{U}(\mathfrak{g})$. This map is then $\mathcal{U}(\mathfrak{b})$-equivariant. We use it to extend the action of $\mathfrak{b}$ on $\mathcal{C}^{\mathrm{pol}}(N, K_\mu)$ to an action of $\mathfrak{g}$. By Lemma 2.5.8 in [8] this action is continuous and $\mathcal{C}^{\mathrm{pol}}(N, K_\mu)$ is a $(\mathfrak{g}, B)$-representation.

We make $\mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(N, K)$ a $(\mathfrak{g}, B)$-module by letting $\mathfrak{g}$ act trivially, N by right translation and T by $tf(n) = f(t^{-1}nt)$ for $t \in T$, $f \in \mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(N, K)$ and $n \in N$. These make the inclusion of $\mathcal{C}_c^{\mathrm{sm}}(N, K)$ into $\mathcal{C}_c^{\mathrm{la}}(N, K)$ $(\mathfrak{g}, B)$-equivariant. We define

$$\mathcal{C}_c^{\mathrm{lp}}(N, K_\mu) = \mathcal{C}^{\mathrm{pol}}(N, K_\mu) \otimes_K \mathcal{C}_c^{\mathrm{sm}}(N, K)$$

with the inductive tensor product topology (this coincides with the projective tensor product topology in this case, by Proposition 1.1.31 of [9]), where "lp" is short for "locally polynomial". We let $\mathfrak{g}$ and B act on both factors. By Lemma 2.5.22 in [8] this action of B is locally analytic. These actions make $\mathcal{C}_c^{\mathrm{lp}}(N, K_\mu)$ a continuous $(\mathfrak{g}, B)$-representation, i.e. the maps $\mathfrak{g} \times \mathcal{C}_c^{\mathrm{lp}}(N, K_\mu) \to \mathcal{C}_c^{\mathrm{lp}}(N, K_\mu)$ and $B \times \mathcal{C}_c^{\mathrm{lp}}(N, K_\mu) \to \mathcal{C}_c^{\mathrm{lp}}(N, K_\mu)$ are both continuous.

Multiplication of algebraic functions by smooth functions gives a map

$$\mathcal{C}_c^{\mathrm{lp}}(N, K_\mu) \longrightarrow \mathcal{C}_c^{\mathrm{la}}(N, K_\mu)$$

which is a continuous, $(\mathfrak{g}, B)$-equivariant injection by Lemma 2.5.24 in [8]. We can think of its image as those locally analytic functions from N to K_μ which are locally given by polynomials.

Now suppose that X is an open subset of N, $\mathbb{X}$ is a rigid analytic affinoid ball defined over L and $\varphi : X \to \mathbb{X}(L)$ is a locally analytic isomorphism compatible with all charts of N. Then the image of the map

$$\mathcal{C}^{\mathrm{pol}}(N, K_\mu) \to \mathcal{C}^{\mathrm{an}}(X, K_\mu) \qquad\qquad f \to f|_X$$

is dense in $\mathcal{C}^{\mathrm{an}}(X, K_\mu)$. It follows that the image of $\mathcal{C}_c^{\mathrm{lp}}(N, K_\mu)$ in $\mathcal{C}_c^{\mathrm{la}}(N, K_\mu)$ is also dense.

Chapter 3

The categories $\mathcal{O}$ and $\overline{\mathcal{O}}$

For this chapter only we change the notation. We work with a semisimple Lie algebra $\mathfrak{g}$ over a field K (not L) with a split Cartan subalgebra $\mathfrak{h}$ (i.e. for all $X \in \mathfrak{h}$ the eigenvalues of $\operatorname{ad} X$ are in K) and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ with nilpotent radical $\mathfrak{n}$. Write $\overline{\mathfrak{b}}$ for the opposite Borel subalgebra to $\mathfrak{b}$. For more background on the category $\mathcal{O}$ see [13].

Let $\mathcal{U}(\mathfrak{g})$-Mod denote the category of $\mathcal{U}(\mathfrak{g})$-modules with morphisms given by morphisms of vector spaces which commute with the $\mathcal{U}(\mathfrak{g})$-actions. Recall S is the principal anti-automorphism of $\mathcal{U}(\mathfrak{g})$. For $M \in \mathcal{U}(\mathfrak{g})\text{-Mod}$ the dual module $M^* \in \mathcal{U}(\mathfrak{g})\text{-Mod}$ has the action given by $u\phi(m) = \phi(S(u)m)$ for any $u \in \mathcal{U}(\mathfrak{g})$, $\phi \in M^*$ and $m \in M$.

Let $\mathcal{C}$ denote the full subcategory of $\mathcal{U}(\mathfrak{g})$-Mod given by those modules M on which $\mathfrak{h}$ acts diagonalisably and the weight spaces are finite dimensional. It is easily checked that this is an abelian category.

If $M \in \mathcal{C}$ and μ is a weight of M then we have an injection $i : (M_\mu)^* \to M^*$ by extending $\phi \in (M_\mu)^*$ by 0 on all the other weight spaces M_ν, and $i((M_\mu)^*) \subset (M^*)_{-\mu}$. Moreover, for any $\phi \in (M^*)_{-\mu}$, $\phi(M_\nu) = 0$ unless $\nu = \mu$, so we have equality: $i((M_\mu)^*) = (M^*)_{-\mu}$.

Since $M = \bigoplus_{\mu\in\mathfrak{h}^*} M_\mu$ we get that $M^* = \prod_{\mu\in\mathfrak{h}^*}(M_\mu)^* = \prod_{\mu\in\mathfrak{h}^*}(M^*)_\mu$. We define $M^\vee$ to be

$$M^\vee = \bigoplus_{\mu\in\mathfrak{h}^*}(M^*)_\mu \subset M^*.$$

Note that the action of $\mathfrak{h}$ preserves $(M^*)_\mu$ and if $\phi \in (M^*)_\mu$ and $X \in \mathfrak{g}_\alpha$ then we have

$X\phi \in (M^*)_{\mu-\alpha}$. It follows that $M^\vee$ is a $\mathcal{U}(\mathfrak{g})$-submodule of M^*.

Clearly $\mathfrak{h}$ acts diagonalisably on $M^\vee$. Since $(M^*)_\mu \cong (M_{-\mu})^*$ the weight spaces are finite dimensional. Hence $M^\vee \in \mathcal{C}$.

Lemma 2. *The contravariant functor $F : \mathcal{C} \to \mathcal{C}$ given by $M \to M^\vee$ is an anti-equivalence of categories. In particular it is exact.*

Proof. First F is a functor as any $\phi : M \to M'$ restricts to a map $M_\mu \to M'_\mu$ for any $\mu \in \mathfrak{h}^*$, and it is contravariant as $M \to M^*$ is. Now consider $(M^\vee)^\vee \subset M^{**}$. It is a direct sum of its weight spaces and the μ weight space is $((M^\vee)_{-\mu})^*$, which is $(M_\mu)^{**}$, precisely the image of M_μ under the canonical embedding $M \to M^{**}$. Thus $M^{\vee\vee}$ is isomorphic to M.

It is a standard result in category theory that F is an anti-equivalence of categories if and only if F is fully faithful and essentially surjective. This follows from the fact that $F^2(M) \cong M$ for any $M \in \mathcal{C}$. □

Definition 3. *Let M be a $\mathcal{U}(\mathfrak{g})$-module. For a Lie subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ we say $\mathfrak{a}$ acts **locally finitely** on $x \in M$ if x is contained in some finite dimensional $\mathcal{U}(\mathfrak{a})$-submodule of M.*

Definition 4. *We define the **category** $\mathcal{O}$ to be the full subcategory of $\mathcal{U}(\mathfrak{g})$-*Mod *consisting of finitely generated modules on which $\mathfrak{n}$ acts locally finitely and $\mathfrak{h}$ acts diagonalisably.*

*We define the **category** $\overline{\mathcal{O}}$ to be the full subcategory of $\mathcal{U}(\mathfrak{g})$-*Mod *consisting of finitely generated modules on which $\overline{\mathfrak{n}}$ acts locally finitely and $\mathfrak{h}$ acts diagonalisably.*

Both categories are closed under finite direct sums, submodules and quotients, and both categories contain all finite dimensional $\mathcal{U}(\mathfrak{g})$-modules.

Lemma 5. *The category $\mathcal{O}$ is a subcategory of $\mathcal{C}$.*

Proof. Suppose $M \in \mathcal{O}$. We know $\mathfrak{h}$ acts diagonalisably on M so we just have to show that the weight spaces are finite dimensional.

Let $X \subset M$ be a finite set which generates M as a $\mathcal{U}(\mathfrak{g})$-module. As $\mathfrak{h}$ acts diagonalisably on M we have $M = \bigoplus_{\mu\in\mathfrak{h}^*} M_\mu$, so we can write each $x \in X$ as a sum of finitely many elements of weight spaces. Replacing each $x \in X$ with the elements thus obtained, we may assume that X consists of weight vectors. As $\mathfrak{n}$ acts locally finitely on M, $\mathcal{U}(\mathfrak{n})x$

is finite dimensional for each $x \in X$. It is also a $\mathcal{U}(\mathfrak{h})$-module, so we may pick a basis of weight vectors for it. Replacing each $x \in X$ by this basis we have a finite set X of weight vectors in M which generates M as a $\mathcal{U}(\overline{\mathfrak{n}})$-module.

Choose an ordering $\{\alpha_1, \cdots, \alpha_r\}$ for Φ^+. Then $\{F_{\alpha_1}, \cdots, F_{\alpha_r}\}$ is a basis for $\overline{\mathfrak{n}}$, so by the Poincaré-Birkhoff-Witt theorem $\{F_{\alpha_1}^{n_1} \cdots F_{\alpha_r}^{n_r} : n_i \geq 0 \ \forall i\}$ is a basis for $\mathcal{U}(\overline{\mathfrak{n}})$. It follows that the set $\{F_{\alpha_1}^{n_1} \cdots F_{\alpha_r}^{n_r} x : n_i \geq 0, x \in X\}$ spans M as a vector space over K. If $x \in X$ has weight μ_x then $F_{\alpha_1}^{n_1} \cdots F_{\alpha_r}^{n_r} x$ has weight $\mu_x - \sum n_i \alpha_i$. As each weight can only be written as a sum of positive roots in a finite number of ways, each weight space M_μ must be finite dimensional. □

Lemma 6. *The category $\overline{\mathcal{O}}$ is a subcategory of $\mathcal{C}$.*

Proof. The proof proceeds as above with $\overline{\mathfrak{n}}$ replaced by $\mathfrak{n}$. □

Recall that $\overline{\mathfrak{n}}^k \subset \mathcal{U}(\overline{\mathfrak{n}})$ denotes $\{F_1 \cdots F_k : F_i \in \overline{\mathfrak{n}} \text{ for all } 1 \leq i \leq k\}$ and $M^{\overline{\mathfrak{n}}^\infty}$ denotes all elements of M which are annihilated by $\overline{\mathfrak{n}}^k$ for some k.

Lemma 7. *If $M \in \mathcal{O}$ then $M^\vee = (M^*)^{\overline{\mathfrak{n}}^\infty}$.*

Proof. For any $\mathcal{U}(\overline{\mathfrak{n}})$-module M let $\overline{\mathfrak{n}}^k M$ denote the smallest subspace of M containing $F_1 \cdots F_k v$ for any $F_1, \cdots, F_k \in \overline{\mathfrak{n}}$ and $v \in M$. Let $\mathcal{U}(\overline{\mathfrak{n}})\overline{\mathfrak{n}}^k$ denote the smallest subspace of $\mathcal{U}(\overline{\mathfrak{n}})$ containing $uF_1 \cdots F_k$ for any $F_1, \cdots, F_k \in \overline{\mathfrak{n}}$ and $u \in \mathcal{U}(\overline{\mathfrak{n}})$ (so $\overline{\mathfrak{n}}^k\mathcal{U}(\overline{\mathfrak{n}}) = \mathcal{U}(\overline{\mathfrak{n}})\overline{\mathfrak{n}}^k$). If we pick an ordering $\{\alpha_1, \cdots, \alpha_r\}$ of Φ^+ then $\{F_{\alpha_1}^{n_1} \cdots F_{\alpha_r}^{n_r} : n_j \geq 0\}$ is a basis for $\mathcal{U}(\overline{\mathfrak{n}})$. As $\mathcal{U}(\overline{\mathfrak{n}})\overline{\mathfrak{n}}^k$ contains $\{F_{\alpha_1}^{n_1} \cdots F_{\alpha_r}^{n_r} : \sum n_j \geq k\}$ it has finite codimension in $\mathcal{U}(\overline{\mathfrak{n}})$.

Let X be a finite set of weight vectors which generates M as a $\mathcal{U}(\overline{\mathfrak{n}})$-module, as constructed in the proof of Lemma 5. If $x \in X$ has weight μ_x then $F_{\alpha_{m_1}} \cdots F_{\alpha_{m_k}} x$ has weight $\mu_x - \sum \alpha_{m_i}$ for any $\alpha_{m_1}, \cdots, \alpha_{m_k} \in \Phi$ (i.e. the order is not important), so for any weight μ we can find $k \in \mathbb{N}$ such that $(\overline{\mathfrak{n}}^k M)_\mu = 0$.

Let $\phi \in M^\vee = \bigoplus_{\mu \in \mathfrak{h}^*} (M^*)_\mu$, say $\phi = \sum \phi_\mu$ where $\phi_\mu \in (M^*)_\mu$ for each μ and the sum is over a finite set I. We can find $k \in \mathbb{N}$ such that $(\overline{\mathfrak{n}}^k M)_{-\mu} = 0$ for all $\mu \in I$, i.e. $\phi|_{\overline{\mathfrak{n}}^k M} = 0$. Then $\overline{\mathfrak{n}}^k \phi(M) = \phi(\overline{\mathfrak{n}}^k M) = 0$, so $\phi \in (M^*)^{\overline{\mathfrak{n}}^\infty}$. Hence $M^\vee \subset (M^*)^{\overline{\mathfrak{n}}^\infty}$.

Now suppose $\phi \in (M^*)^{\overline{\mathfrak{n}}^\infty}$. Choose k such that $\overline{\mathfrak{n}}^k \phi = 0$. Then $\mathcal{U}(\overline{\mathfrak{n}})\overline{\mathfrak{n}}^k x \subset \ker \phi$ for all $x \in X$. The action of $\mathfrak{h}$ preserves $\sum_{x \in X} \mathcal{U}(\overline{\mathfrak{n}})\overline{\mathfrak{n}}^k x \subset M$, so it splits up into weight spaces. If we can show that the number of μ such that $M_\mu \not\subseteq \sum_{x \in X} \mathcal{U}(\overline{\mathfrak{n}})\overline{\mathfrak{n}}^k x$ is finite then ϕ can only be non-zero on finitely many M_μ, and hence $\phi \in \bigoplus_{\mu\in\mathfrak{h}^*}(M^*)_\mu$. This would prove that $(M^*)^{\overline{\mathfrak{n}}^\infty} \subset M^\vee$.

If we have $y \in M_\mu$ then as $M = \sum_{x\in X}\mathcal{U}(\overline{\mathfrak{n}})x$ we can write $y = \sum u_x x$ with each $u_x \in \mathcal{U}(\overline{\mathfrak{n}})$. As each $x \in X$ is a weight vector, $\mathfrak{h}$ acts diagonalisably on $\mathcal{U}(\overline{\mathfrak{n}})x$, so $\mathcal{U}(\overline{\mathfrak{n}})x = \bigoplus_{\mu\in\mathfrak{h}^*}(\mathcal{U}(\overline{\mathfrak{n}})x)_\mu$. We may thus replace each $u_x x$ with its component in $(\mathcal{U}(\overline{\mathfrak{n}})x)_\mu$ and still get $y = \sum u_x x$ with each $u_x x$ of weight μ. Thus we need to show that for each $x \in X$ there are only finitely many μ such that $(\mathcal{U}(\overline{\mathfrak{n}})x)_\mu \not\subseteq \mathcal{U}(\overline{\mathfrak{n}})\overline{\mathfrak{n}}^k x$. This follows from the fact that $\mathcal{U}(\overline{\mathfrak{n}})\overline{\mathfrak{n}}^k$ has finite codimension in $\mathcal{U}(\overline{\mathfrak{n}})$. □

Lemma 8. *If $M \in \overline{\mathcal{O}}$ then $M^\vee = (M^*)^{\mathfrak{n}^\infty}$.*

Proof. The proof proceeds as above with $\overline{\mathfrak{n}}$ replaced by $\mathfrak{n}$. □

Lemma 9. *If $M \in \mathcal{O}$ then $M^\vee \in \overline{\mathcal{O}}$.*

Proof. As $M^\vee \in \mathcal{C}$ the action of $\mathfrak{h}$ on it is diagonalisable. We showed in Lemma 7 that $M^\vee = (M^*)^{\overline{\mathfrak{n}}^\infty}$. This implies that $\overline{\mathfrak{n}}$ acts locally finitely on $M^\vee$: if $\phi \in M^\vee$ then we can find k such that $\overline{\mathfrak{n}}^k$ annihilates ϕ, so the surjection from $\mathcal{U}(\overline{\mathfrak{n}})$ to $\mathcal{U}(\overline{\mathfrak{n}})\phi$ factors through $\mathcal{U}(\overline{\mathfrak{n}})/(\mathcal{U}(\overline{\mathfrak{n}})\overline{\mathfrak{n}}^k)$, which is finite dimensional. It only remains to show that $M^\vee$ is finitely generated.

As $M \in \mathcal{O}$ it has finite length (cf. §1.11 in [13]). If we can show that $M^\vee$ has finite length then it follows that it is finitely generated. We proceed by induction on the length of M. Suppose first that M is a simple object in $\mathcal{O}$. Then M is a simple object in $\mathcal{C}$, and so is $M^\vee$ by Lemma 2.

Now suppose M has length $k > 1$ and let M_0 be a maximal proper submodule, which must have length $k-1$. Applying the exact functor $\mathcal{C} \to \mathcal{C}$ from Lemma 2 to the exact sequence

$$0 \to M_0 \to M \to M/M_0 \to 0$$

we get an exact sequence

$$0 \to (M/M_0)^\vee \to M^\vee \to M_0^\vee \to 0$$

where the first term is simple and the last term has finite length, by the inductive hypothesis. It follows that $M^\vee$ has finite length, and in fact that it has the same length as M. □

Lemma 10. *If $M \in \overline{\mathcal{O}}$ then $M^{\vee} \in \mathcal{O}$.*

Proof. The proof proceeds as above with $\overline{\mathfrak{n}}$ replaced by $\mathfrak{n}$. □

It follows that $F : \overline{\mathcal{O}} \to \mathcal{O}$ and $F' : \mathcal{O} \to \overline{\mathcal{O}}$ both given by $M \mapsto M^{\vee}$ give an anti-equivalence of categories. In particular they are both exact functors.

Chapter 4

Constructing maps between the spaces $\mathcal{C}^{\mathbf{la}}_{\mathbf{c}}(N, K_\mu)$

For $\mu \in \mathfrak{h}^*$ let K_μ denote a one-dimensional K-vector space with the action of $\mathcal{U}(\mathfrak{b})$ given by extending μ to $\mathfrak{b}$ by letting $\mathfrak{n}$ act trivially. For $\mu \in X(\mathbf{T})$ this is consistent with our earlier definition. We define $M_{\mathfrak{b}}(\mu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} K_\mu$, where $\mathcal{U}(\mathfrak{g})$ is a $\mathcal{U}(\mathfrak{b})$-module by multiplication on the right. Then $M_{\mathfrak{b}}(\mu)$ is a Verma module for the Borel subalgebra $\mathfrak{b}$. It is a highest weight module, with highest weight μ. It is in $\mathcal{O}$, and hence in $\mathcal{C}$ by Lemma 5.

Similarly, let A_μ be a one-dimensional K-vector space with the action of $\mathcal{U}(\overline{\mathfrak{b}})$ given by extending μ to $\overline{\mathfrak{b}}$ by letting $\overline{\mathfrak{n}}$ act trivially. Define

$$M_{\overline{\mathfrak{b}}}(\mu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{b}})} A_\mu^*.$$

This is a Verma module for $\overline{\mathfrak{b}}$. It is a lowest weight module, with lowest weight $-\mu$. It is in $\overline{\mathcal{O}}$, and hence in $\mathcal{C}$ by Lemma 6. For reasons that will become clear in Chapter 5 we are more interested in $M_{\overline{\mathfrak{b}}}(\mu)$ than in $M_{\mathfrak{b}}(\mu)$.

Suppose we have a fixed $\mathcal{U}(\mathfrak{g})$-equivariant morphism

$$\psi : M_{\overline{\mathfrak{b}}}(\mu_2) \to M_{\overline{\mathfrak{b}}}(\mu_1)$$

for some μ_1 and μ_2 in $X(\mathbf{T})$. Applying the functor $F : \overline{\mathcal{O}} \to \mathcal{O}$, $M \to M^\vee$ we get a map

$$\psi^\vee : M_{\overline{\mathfrak{b}}}(\mu_1)^\vee \to M_{\overline{\mathfrak{b}}}(\mu_2)^\vee$$

of $\mathcal{U}(\mathfrak{g})$-modules.

Proposition 5.5.4 of [7] says that for any finite-dimensional $\mathcal{U}(\overline{\mathfrak{b}})$-module M the map

$$(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{b}})} M)^* \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}(\overline{\mathfrak{b}})}(\mathcal{U}(\mathfrak{g}), M^*) \qquad \varphi \mapsto (u \mapsto (m \mapsto \varphi(u \otimes m)))$$

is an isomorphism of $\mathcal{U}(\mathfrak{g})$-modules. Here $\mathcal{U}(\mathfrak{g})$ is a $\mathcal{U}(\overline{\mathfrak{b}})$-module by the left regular representation and $\mathcal{U}(\mathfrak{g})$ acts on $\mathrm{Hom}_{\mathcal{U}(\overline{\mathfrak{b}})}(\mathcal{U}(\mathfrak{g}), M^*)$ by multiplication on the right on the source, i.e. $u\phi(u') = \phi(u'u)$ for all $u, u' \in \mathcal{U}(\mathfrak{g})$ and $\phi \in \mathrm{Hom}_{\mathcal{U}(\overline{\mathfrak{b}})}(\mathcal{U}(\mathfrak{g}), M^*)$. Setting $M = A_\mu$ and using the natural isomorphism $A_\mu^{**} \cong A_\mu$, we get $M_{\overline{\mathfrak{b}}}(\mu)^* \cong \mathrm{Hom}_{\mathcal{U}(\overline{\mathfrak{b}})}(\mathcal{U}(\mathfrak{g}), A_\mu)$ as $\mathcal{U}(\mathfrak{g})$-modules. Now let $\mu \in X(\mathbf{T})$. Combining this isomorphism with Lemma 8 and the $\mathcal{U}(\mathfrak{g})$-equivariant isomorphism $\mathcal{C}^{pol}(N, K_\mu) \cong \mathrm{Hom}_{\mathcal{U}(\overline{\mathfrak{b}})}(\mathcal{U}(\mathfrak{g}), A_\mu)^{\mathfrak{n}^\infty}$ from Chapter 2 we get an isomorphism of $\mathcal{U}(\mathfrak{g})$-modules

$$\zeta_\mu : M_{\overline{\mathfrak{b}}}(\mu)^\vee \xrightarrow{\sim} \mathcal{C}^{pol}(N, K_\mu).$$

Following all the definitions we see that for any $u \in \mathcal{U}(\mathfrak{g})$ and $\varphi \in M_{\overline{\mathfrak{b}}}(\mu)^\vee$ we have $u\zeta_\mu(\varphi)(e) = \varphi(S(u) \otimes 1)$, or equivalently $S(u)\zeta_\mu(\varphi)(e) = \varphi(u \otimes 1)$.

We define

$$\psi^{pol} : \mathcal{C}^{pol}(N, K_{\mu_1}) \to \mathcal{C}^{pol}(N, K_{\mu_2})$$

by $\psi^{pol} = \zeta_{\mu_2} \circ \psi^\vee \circ \zeta_{\mu_1}^{-1}$. This is a morphism of $\mathcal{U}(\mathfrak{g})$-modules. Recall that $\mathcal{C}_c^{lp}(N, K_\mu) = \mathcal{C}^{pol}(N, K_\mu) \otimes_K \mathcal{C}_c^{sm}(N, K)$. We define $\psi^{lp} : \mathcal{C}_c^{lp}(N, K_{\mu_1}) \longrightarrow \mathcal{C}_c^{lp}(N, K_{\mu_2})$ on simple tensors by

$$\psi^{lp}(f_{pol} \otimes f_{sm}) = \psi^{pol}(f_{pol}) \otimes f_{sm}$$

and extend K-linearly. This is $\mathcal{U}(\mathfrak{g})$-equivariant, as $\mathfrak{g}$ acts trivially on $\mathcal{C}_c^{sm}(N, K)$.

We spend the remainder of this chapter proving there is a unique continuous map $\psi^{la} : \mathcal{C}_c^{la}(N, K_{\mu_1}) \to \mathcal{C}_c^{la}(N, K_{\mu_2})$ extending ψ^{lp} and that it is $(\mathfrak{g}, B)$-equivariant.

By the Poincaré-Birkhoff-Witt theorem there is a unique $u_\psi \in \mathcal{U}(\mathfrak{n})$ which satisfies

$\psi(1 \otimes 1) = u_\psi \otimes 1$. Since $M_{\overline{\mathfrak{b}}}(\mu_2) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{b}})} A^*_{\mu_2}$ is generated as a $\mathcal{U}(\mathfrak{g})$-module by the single element $1 \otimes 1$, u_ψ determines ψ by the formula $\psi(u \otimes 1) = u(\psi(1 \otimes 1)) = uu_\psi \otimes 1$.

Since $\zeta_\mu^{-1} : \mathcal{C}^{\mathrm{pol}}(N, K_\mu) \to M_{\overline{\mathfrak{b}}}(\mu)^\vee$ sends f to the map $u \otimes 1 \mapsto S(u)f(e)$, for any $f \in \mathcal{C}^{\mathrm{pol}}(N, K_{\mu_1})$ we have

$$S(u)\psi^{\mathrm{pol}}(f)(e) = S(uu_\psi)f(e) \tag{4.1}$$

for all $u \in \mathcal{U}(\mathfrak{g})$. Moreover, since ζ_{μ_2} is an isomorphism any $f' \in \mathcal{C}^{\mathrm{pol}}(N, K_{\mu_2})$ is determined by knowing $S(u)f'(e)$ for all $u \in \mathcal{U}(\mathfrak{g})$, so (4.1) uniquely determines $\psi^{\mathrm{pol}}(f)$.

Let us examine the $\mathcal{U}(\mathfrak{g})$-action on $\mathcal{C}^{\mathrm{lp}}(N, K_{\mu_1})$. By Lemma 2.5.24 of [8], the natural map $\mathcal{C}^{\mathrm{lp}}_{\mathrm{c}}(N, K_{\mu_1}) \to \mathcal{C}^{\mathrm{la}}_{\mathrm{c}}(N, K_{\mu_1})$ given by multiplication of polynomial functions by smooth functions is a $\mathcal{U}(\mathfrak{g})$-equivariant injection. The $\mathcal{U}(\mathfrak{g})$-action on $\mathcal{C}^{\mathrm{la}}_{\mathrm{c}}(N, K_{\mu_1})$ is given by the isomorphism $\mathcal{C}^{\mathrm{la}}_{\mathrm{c}}(N, K_{\mu_1}) \to \mathrm{Ind}_B^G(\mu_1)(N)$. The $\mathcal{U}(\mathfrak{g})$-action on $\mathrm{Ind}_B^G(\mu_1)(N)$ comes from differentiating the right regular action of G on $\mathcal{C}^{\mathrm{la}}(G, K)$.

But there is also the left regular action of G on $\mathcal{C}^{\mathrm{la}}(G, K)$, which we denote with a subscript L. Recall that to make this a left action rather than a right action we define it as

$$h_L f(g) = f(h^{-1}g).$$

We can also differentiate the left regular action of G to get an action of $\mathfrak{g}$, which we call the L action to distinguish it from our original action of $\mathfrak{g}$. Since the left and right regular actions of G on $\mathcal{C}^{\mathrm{la}}(G, K)$ commute, the L action of $\mathfrak{g}$ commutes with the right regular action of G.

For any $g \in G$ and $f \in \mathcal{C}^{\mathrm{la}}(G, K)$ we have

$$(g_L f)(e) = f(g^{-1}) = (g^{-1}f)(e)$$

so for any $X \in \mathfrak{g}$ we have $X_L f(e) = (-X)f(e)$, and hence $u_L f(e) = S(u)f(e)$ for any

$u \in \mathcal{U}(\mathfrak{g})$. It follows that for $u \in \mathcal{U}(\mathfrak{g})$ and $f \in \mathcal{C}^{\mathrm{la}}(G, K)$ we have

$$\begin{aligned} S(u)((u_\psi)_L f)(e) &= (u_\psi)_L(S(u)f)(e) \\ &= S(u_\psi)(S(u)f)(e) \\ &= S(uu_\psi)f(e) \end{aligned} \tag{4.2}$$

which closely resembles (4.1).

We now establish some properties of the L action of $\mathcal{U}(\mathfrak{g})$. Let $\Omega \subseteq G$ be closed and open, $f \in \mathcal{C}^{\mathrm{la}}(G, K)$ and $u \in \mathcal{U}(\mathfrak{g})$. Recall that $f_{|\Omega} = f\mathbf{1}_\Omega$.

Lemma 11. *We have $u_L(f_{|\Omega}) = (u_L f)_{|\Omega}$.*

Proof. For any $X \in \mathfrak{g}$, $X_L(f\mathbf{1}_\Omega) = (X_L f)\mathbf{1}_\Omega + f(X_L\mathbf{1}_\Omega)$, by the Leibniz rule. But $X_L\mathbf{1}_\Omega = 0$ as $\mathbf{1}_\Omega$ is smooth, so $X_L(f\mathbf{1}_\Omega) = (X_L f)\mathbf{1}_\Omega$. It follows that $u_L(f\mathbf{1}_\Omega) = (u_L f)\mathbf{1}_\Omega$ for all $u \in \mathcal{U}(\mathfrak{g})$, and hence that $u_L(f_{|\Omega}) = (u_L f)_{|\Omega}$. □

Corollary 12. *If $g = u_L f$ then $\operatorname{Supp} g \subseteq \operatorname{Supp} f$, and we can find $f' \in \mathcal{C}^{la}(G, K)$ such that $u_L f' = g$ and $\operatorname{Supp} g = \operatorname{Supp} f'$.*

Proof. Since $f = f_{|\operatorname{Supp} f}$ we have that $g = u_L f = u_L(f_{|\operatorname{Supp} f}) = (u_L f)_{|\operatorname{Supp} f} = g_{|\operatorname{Supp} f}$, whence it follows that $\operatorname{Supp} g \subseteq \operatorname{Supp} f$.

Set $f' = f_{|\operatorname{Supp} g}$, so by construction $\operatorname{Supp} f' = \operatorname{Supp} g$. Then $u_L f' = u_L(f_{|\operatorname{Supp} g}) = (u_L f)_{|\operatorname{Supp} g} = g_{|\operatorname{Supp} g} = g$. □

Suppose further that there is a locally analytically isomorphism between Ω and the L-points of a rigid analytic space which is compatible with all charts of G.

Lemma 13. *If f is analytic on Ω then $(u_L f)$ is also analytic on Ω.*

Proof. The L action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{C}^{\mathrm{la}}(G, K)$, and hence on $\mathcal{C}^{\mathrm{an}}(\Omega, K)$, is via differential operators, which preserve analytic functions. □

Now we are in a position to define ψ^{la}.

Theorem 14. *There is a unique continuous map $\psi^{la} : \mathcal{C}_c^{la}(N, K_{\mu_1}) \to \mathcal{C}_c^{la}(N, K_{\mu_2})$ extending ψ^{lp}, and moreover it is $(\mathfrak{g}, B)$-equivariant.*

Proof. Uniqueness is immediate by the density of $\mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{\mu_1})$ in $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_{\mu_1})$.

Let $f_{\mathrm{pol}} \in \mathcal{C}^{\mathrm{pol}}(N, K_{\mu_1})$ and $f_{\mathrm{sm}} \in \mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(N, K)$. We identify $\mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_\mu)$ with its image in $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)$ and we define

$$\Phi_\mu : \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu) \longrightarrow \mathcal{C}^{\mathrm{la}}(G, K)$$

to be the continuous, $(\mathfrak{g}, B)$-equivariant injection obtained by composing the isomorphism $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu) \cong \mathrm{Ind}_B^G(\mu)(N)$ with the inclusion $\mathrm{Ind}_B^G(\mu)(N) \subseteq \mathcal{C}^{\mathrm{la}}(G, K)$. This means that $\Phi_\mu(f_{\mathrm{pol}} \otimes f_{\mathrm{sm}})$ is defined on $\overline{B}N$ by $\bar{b}n \mapsto \mu(\bar{b}) f_{\mathrm{pol}}(n) f_{\mathrm{sm}}(n)$ and is 0 outside of $\overline{B}N$. In particular, $\Phi_\mu(f_{\mathrm{pol}} \otimes f_{\mathrm{sm}})(e) = f_{\mathrm{pol}}(e) f_{\mathrm{sm}}(e)$.

Let us examine $F = (u_\psi)_L \Phi_{\mu_1}(f_{\mathrm{pol}} \otimes f_{\mathrm{sm}}) - \Phi_{\mu_2}(\psi^{\mathrm{lp}}(f_{\mathrm{pol}} \otimes f_{\mathrm{sm}}))$. By (4.1), (4.2) and the $\mathcal{U}(\mathfrak{g})$-equivariance of Φ_μ, for all $u \in \mathcal{U}(\mathfrak{g})$ we have

$$\begin{aligned} S(u)F(e) &= S(u)\left((u_\psi)_L(\Phi_{\mu_1}(f_{\mathrm{pol}} \otimes f_{\mathrm{sm}}))\right)(e) - S(u)(\Phi_{\mu_2}(\psi^{\mathrm{pol}}(f_{\mathrm{pol}}) \otimes f_{\mathrm{sm}}))(e) \\ &= S(uu_\psi)(\Phi_{\mu_1}(f_{\mathrm{pol}} \otimes f_{\mathrm{sm}}))(e) - \Phi_{\mu_2}(S(u)\psi^{\mathrm{pol}}(f_{\mathrm{pol}}) \otimes f_{\mathrm{sm}})(e) \\ &= \Phi_{\mu_1}(S(uu_\psi) f_{\mathrm{pol}} \otimes f_{\mathrm{sm}})(e) - S(u)\psi^{\mathrm{pol}}(f_{\mathrm{pol}})(e) f_{\mathrm{sm}}(e) \\ &= S(uu_\psi) f_{\mathrm{pol}}(e) f_{\mathrm{sm}}(e) - S(uu_\psi) f_{\mathrm{pol}}(e) f_{\mathrm{sm}}(e) \\ &= 0. \end{aligned}$$

We have just shown that $uF(e) = 0$ for all $u \in \mathcal{U}(\mathfrak{g})$, and hence that the image of F under all point distributions at e is 0. It follows that F must be identically 0 on some neighbourhood of e.

Let X be a chart of N containing e and set $f_{\mathrm{sm}} = \mathbf{1}_X$. Then $F \in \mathcal{C}^{\mathrm{la}}(G, K)$ has $\mathrm{Supp}\, F \subseteq \overline{B}X$ by Corollary 12, and it is analytic on $\overline{B}X$ by Lemma 13. Hence it is 0 on $\overline{B}X$, and we have shown that $F = 0$.

Let $Y \subseteq N$ be any compact, open subset, and choose a chart $X \subseteq N$ containing e such that $Y \subseteq X$. By the above argument we know that $\Phi_{\mu_2}(\psi^{\mathrm{lp}}(f_{\mathrm{pol}} \otimes \mathbf{1}_X)) = (u_\psi)_L \Phi_{\mu_1}(f_{\mathrm{pol}} \otimes$

$\mathbf{1}_X$). Then, using Lemma 11 and the fact that $\Phi_{\mu_2}(g_{|Y}) = \Phi_{\mu_2}(g)_{|\overline{B}Y}$, we have that

$$\begin{aligned}
\Phi_{\mu_2}(\psi^{\mathrm{lp}}(f_{\mathrm{pol}} \otimes \mathbf{1}_Y)) &= \Phi_{\mu_2}((\psi^{\mathrm{pol}}(f_{\mathrm{pol}}) \otimes \mathbf{1}_X)_{|Y}) \\
&= (\Phi_{\mu_2}(\psi^{\mathrm{pol}}(f_{\mathrm{pol}}) \otimes \mathbf{1}_X))_{|\overline{B}Y} \\
&= ((u_\psi)_L \Phi_{\mu_1}(f_{\mathrm{pol}} \otimes \mathbf{1}_X))_{|\overline{B}Y} \\
&= (u_\psi)_L (\Phi_{\mu_1}(f_{\mathrm{pol}} \otimes \mathbf{1}_X)_{|\overline{B}Y}) \\
&= (u_\psi)_L \Phi_{\mu_1}((f_{\mathrm{pol}} \otimes \mathbf{1}_X)_{|Y}) \\
&= (u_\psi)_L \Phi_{\mu_1}(f_{\mathrm{pol}} \otimes \mathbf{1}_Y).
\end{aligned}$$

By linearity it follows that $\Phi_{\mu_2}(\psi^{\mathrm{lp}}(f)) = (u_\psi)_L \Phi_{\mu_1}(f)$ for all $f \in \mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{\mu_1})$. From this we may deduce that

$$\begin{aligned}
(u_\psi)_L \Phi_{\mu_1}(\mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{\mu_1})) &= \Phi_{\mu_2}(\psi^{\mathrm{lp}}(\mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{\mu_1}))) \\
&\subseteq \Phi_{\mu_2}(\mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{\mu_2})) \\
&\subseteq \mathrm{Ind}_{\overline{B}}^{G}(\mu_2)(N).
\end{aligned}$$

All the maps involved are continuous, $\mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{\mu_1})$ is dense in $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_{\mu_1})$ and $\mathrm{Ind}_{\overline{B}}^{G}(\mu_2)(N)$ is a closed subspace of $\mathcal{C}^{\mathrm{la}}(G, K)$, so it follows that $(u_\psi)_L \Phi_{\mu_1}(\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_{\mu_1})) \subseteq \mathrm{Ind}_{\overline{B}}^{G}(\mu_2)(N)$.

Since $\Phi_{\mu_2} : \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_{\mu_2}) \to \mathrm{Ind}_{\overline{B}}^{G}(\mu_2)(N)$ is an isomorphism we can define

$$\psi^{\mathrm{la}} = \Phi_{\mu_2}^{-1} \circ (u_\psi)_L \circ \Phi_{\mu_1} : \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_{\mu_1}) \to \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_{\mu_2}).$$

This is continuous and $(\mathfrak{g}, B)$-equivariant as all the maps involved in its definition are. As $\Phi_{\mu_2}(\psi^{\mathrm{lp}}(f)) = (u_\psi)_L \Phi_{\mu_1}(f)$ for all $f \in \mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{\mu_1})$, this extends ψ^{lp}. □

Lemma 15. *Let* $f \in \mathcal{C}_c^{la}(N, K_{\mu_1})$.

1. *If* $\Omega \subseteq N$ *is open and closed then* $\psi^{la}(f_{|\Omega}) = \psi^{la}(f)_{|\Omega}$.
2. $\mathrm{Supp}\, \psi^{la}(f) \subseteq \mathrm{Supp}\, f$.
3. $f' = f_{|\mathrm{Supp}\, \psi^{la}(f)}$ *satisfies* $\psi^{la}(f') = \psi^{la}(f)$ *and* $\mathrm{Supp}\, f' = \mathrm{Supp}\, \psi^{la}(f)$.
4. *If* f *is analytic on* $X \subseteq N$ *then so is* $\psi^{la}(f)$.

5. *ψ^{pol} and ψ^{lp} are $(\mathfrak{g}, B)$-equivariant.*

Proof. Parts 1-4 follow immediately from Lemma 11, Corollary 12 and Lemma 13. Part 5 follows from the fact that ψ^{la} extends ψ^{lp} and

$$\begin{aligned}\psi^{\mathrm{pol}}(bf_{\mathrm{pol}}) \otimes f_{\mathrm{sm}} &= \psi^{\mathrm{lp}}(b(f_{\mathrm{pol}} \otimes b^{-1} f_{\mathrm{sm}})) = b\psi^{\mathrm{lp}}(f_{\mathrm{pol}} \otimes b^{-1} f_{\mathrm{sm}}) \\ &= b(\psi^{\mathrm{pol}}(f_{\mathrm{pol}}) \otimes b^{-1} f_{\mathrm{sm}}) = b\psi^{\mathrm{pol}}(f_{\mathrm{pol}}) \otimes f_{\mathrm{sm}}\end{aligned}$$

for all $f_{\mathrm{pol}} \in \mathcal{C}^{\mathrm{pol}}(N, K_{\mu_1})$, $f_{\mathrm{sm}} \in \mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(N, K)$ and $b \in B$. □

Chapter 5

A BGG-type resolution

The aim of this chapter is to define maps $\theta^{\mathrm{la}}_{\alpha,w\cdot\lambda} : \mathcal{C}^{\mathrm{la}}_{\mathrm{c}}(N, K_{w\cdot\lambda}) \to \mathcal{C}^{\mathrm{la}}_{\mathrm{c}}(N, K_{s_\alpha w\cdot\lambda})$ and construct an exact sequence using them.

Let us first assume that $\mathbf{G}$ is semisimple, so that we can apply results from Chapter 3 to the semisimple Lie algebra $\mathfrak{g}_K = \mathrm{Lie}(\mathbf{G}(K))$, Cartan subalgebra $\mathfrak{h}_K = \mathrm{Lie}(\mathbf{T}(K))$ (which is split because $\mathbf{T}$ is maximal and split over K) and Borel subalgebra $\mathfrak{b}_K = \mathrm{Lie}(\mathbf{B}(K))$. Since $\mathfrak{g}_K = \mathfrak{g} \otimes_L K$, representations of $\mathfrak{g}$ over K are exactly the same thing as representations of $\mathfrak{g}_K$ over K, and we will implicitly equate the two.

It is well-known that W is finite and is generated by the reflections $\{s_\alpha : \alpha \in \Delta\}$. We define the length function l on W by setting $l(w)$ to be the minimal length of a word for w in the $\{s_\alpha : \alpha \in \Delta\}$. We write $W^{(i)}$ for $\{w \in W : l(w) = i\}$, so $W^{(0)} = \{e\}$ and $W^{(1)}$ is $\{s_\alpha : \alpha \in \Delta\}$. The maximal length is $r = |\Phi^+|$ and there is a unique element of length r, which we denote w_0.

Let $\lambda \in X(\mathbf{T})$ be a dominant weight and let $\sigma : \mathbf{G} \to \mathbf{GL}_s$ be the irreducible finite-dimensional representation of $\mathbf{G}$ with highest weight λ . As $\mathbf{T}$ is split over K, σ is defined over K. Let V denote K^s with the action of G given by $\sigma : G \to \mathbf{GL}_s(K)$. The dual representation V^* is then a finite dimensional irreducible algebraic representation of G over K, with lowest weight $-\lambda$.

The **Bernstein-Gelfand-Gelfand resolution** of V^* with respect to $\overline{\mathfrak{b}}$ is the exact sequence of $\mathcal{U}(\mathfrak{g})$-modules

$$0 \to M_{\overline{\mathfrak{b}}}(w_0 \cdot \lambda) \to \cdots \to \bigoplus_{w \in W^{(i)}} M_{\overline{\mathfrak{b}}}(w \cdot \lambda) \to \cdots \to \bigoplus_{w \in W^{(1)}} M_{\overline{\mathfrak{b}}}(w \cdot \lambda) \to M_{\overline{\mathfrak{b}}}(\lambda) \to V^* \to 0. \quad (5.1)$$

The BGG resolution was first constructed in [1] for a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. A more recent treatment is given in [13]. As indicated at the beginning of §0.1 of [13], $\mathbb{C}$ is normally taken to be the field for convenience, but all that is required is that the field K has characteristic 0 and $\mathfrak{h}$ is a split Cartan subalgebra over K. It can be checked that the proof of the BGG resolution given in [13] holds in this case.

With the exception of $M_{\overline{\mathfrak{b}}}(\lambda) \to V^*$, the maps in (5.1) are of the form

$$\bigoplus_{w' \in W^{(i)}} M_{\overline{\mathfrak{b}}}(w' \cdot \lambda) \longrightarrow \bigoplus_{w \in W^{(i-1)}} M_{\overline{\mathfrak{b}}}(w \cdot \lambda)$$

$$(f_{w'})_{w' \in W^{(i)}} \longmapsto \left(\sum_{\substack{\alpha \in \Phi^+ \\ l(s_\alpha w) = i}} \theta_{\alpha, w\cdot\lambda}(f_{s_\alpha w}) \right)_{w \in W^{(i-1)}}$$

where $\theta_{\alpha,w\cdot\lambda}$ denotes a non-zero map $M_{\overline{\mathfrak{b}}}(s_\alpha w \cdot \lambda) \to M_{\overline{\mathfrak{b}}}(w \cdot \lambda)$. Using the results of Chapter 4 with $\psi = \theta_{\alpha,w\cdot\lambda}$ we define $\theta^\vee_{\alpha,w\cdot\lambda}$, $\theta^{\text{pol}}_{\alpha,w\cdot\lambda}$, $\theta^{\text{lp}}_{\alpha,w\cdot\lambda}$ and $\theta^{\text{la}}_{\alpha,w\cdot\lambda}$.

Since all objects and morphisms in (5.1) are in $\overline{\mathcal{O}}$, we can apply the contravariant exact functor $F : \overline{\mathcal{O}} \to \mathcal{O}, M \mapsto M^\vee$ to get the following exact sequence in $\mathcal{O}$:

$$0 \to V \to M_{\overline{\mathfrak{b}}}(\lambda)^\vee \to \bigoplus_{w \in W^{(1)}} M_{\overline{\mathfrak{b}}}(w \cdot \lambda)^\vee \to \cdots \to \bigoplus_{w \in W^{(i)}} M_{\overline{\mathfrak{b}}}(w \cdot \lambda)^\vee \to \cdots \to M_{\overline{\mathfrak{b}}}(w_0 \cdot \lambda)^\vee \to 0 \quad (5.2)$$

Using the isomorphisms $\zeta_{w\cdot\lambda} : M_{\overline{\mathfrak{b}}}(w\cdot\lambda)^\vee \xrightarrow{\sim} \mathcal{C}^{\mathrm{pol}}(N, K_{w\cdot\lambda})$ we rewrite (5.2) as

$$0 \to V \to \mathcal{C}^{\mathrm{pol}}(N, K_\lambda) \to \bigoplus_{w\in W^{(1)}} \mathcal{C}^{\mathrm{pol}}(N, K_{w\cdot\lambda}) \to \cdots \to \bigoplus_{w\in W^{(i)}} \mathcal{C}^{\mathrm{pol}}(N, K_{w\cdot\lambda}) \to \cdots \to \mathcal{C}^{\mathrm{pol}}(N, K_{w_0\cdot\lambda}) \to 0 \quad (5.3)$$

We now remove the assumption that $\mathbf{G}$ is semisimple.

Theorem 16. *We have the exact sequence of* $\mathcal{U}(\mathfrak{g})$*-modules* (5.3) *when* $\mathbf{G}$ *is reductive.*

Proof. Let $\mathbf{G}'$ denote the derived subgroup of $\mathbf{G}$, which is defined over L by the first Corollary in §2.3 of [2]. Note that $\mathbf{G}'$ is semisimple and $\mathbf{T}' = \mathbf{T} \cap \mathbf{G}'$ is a maximal torus in $\mathbf{G}'$ and splits over K. Let $\mathbf{Z}$ denote the center of $\mathbf{G}$. It is defined over L by 12.1.7(b) of [22]. Write Z for $\mathbf{Z}(L)$ and $\mathfrak{z}$ for the corresponding Lie subalgebra of $\mathfrak{g}$. Since $\mathbf{G} = \mathbf{G}'\mathbf{Z}$ it follows that $\mathfrak{g} = \mathfrak{g}' + \mathfrak{z}$. Recall that W is the quotient of the normaliser $\mathbf{N_G}(\mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$ by the centraliser $\mathbf{C_G}(\mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$. Since $\mathbf{Z}$ centralises $\mathbf{T}$, the Weyl groups for $\mathbf{G}'$ and $\mathbf{G}$ are canonically isomorphic. As $\mathbf{N} = \mathbf{N} \cap \mathbf{G}'$, (5.3) for $\mathbf{G}'$ almost gives us the required exact sequence. The problem is that we only know that the maps are $\mathcal{U}(\mathfrak{g}')$-equivariant, where $\mathfrak{g}' = \mathrm{Lie}(G')$. As $\mathfrak{g} = \mathfrak{g}' + \mathfrak{z}$, it suffices to show that they are also $\mathcal{U}(\mathfrak{z})$-equivariant.

The action of T on a highest weight vector v of V is via λ. Since V is an irreducible representation of G, the set $\{gv : g \in G\}$ spans V over K. Since Z is contained in the centre of G we have $zgv = gzv = \lambda(z)gv$, so the action of Z on gv is via λ. Hence Z acts on all of V via λ.

Let us now consider the action of Z on $\mathcal{C}^{\mathrm{pol}}(N, K_{w\cdot\lambda})$. For $z \in Z$, $x \in N$ and $f \in \mathcal{C}^{\mathrm{pol}}(N, K_{w\cdot\lambda})$, since $Z \subseteq T$ we have

$$(zf)(g) = (w\cdot\lambda)(z)f(z^{-1}gz) = (w\cdot\lambda)(z)f(g).$$

So Z acts through $w\cdot\lambda : T \to K^\times$, and hence $\mathfrak{z}$ acts through $w\cdot\lambda \in \mathfrak{h}^*$. The action of W on $X(\mathbf{T})$ comes from the conjugation action of $\mathbf{N_G}(\mathbf{T})$ on $\mathbf{T}$. This action is trivial on $\mathbf{Z} \subseteq \mathbf{T}$, so $\lambda|_Z = (w\cdot\lambda)|_Z$ for all $w \in W$, and hence all the maps in the sequence are $\mathcal{U}(\mathfrak{z})$-equivariant. □

We now tensor (5.3) over K with $\mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(N, K)$. This preserves exactness, as any module over a field is flat and exactness is a property only of the underlying sequence of vector spaces. Thus we get the exact sequence of $\mathcal{U}(\mathfrak{g})$-modules:

$$\begin{aligned} 0 \to V \otimes_K \mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(N, K) \to \mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_\lambda) \to \bigoplus_{w \in W^{(1)}} \mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{w \cdot \lambda}) \to \\ \cdots \to \bigoplus_{w \in W^{(i)}} \mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{w \cdot \lambda}) \to \cdots \to \mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{w_0 \cdot \lambda}) \to 0 \end{aligned} \tag{5.4}$$

With the exception of $V \otimes_K \mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(N, K) \to \mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_\lambda)$, the maps in (5.4) are of the form

$$\begin{aligned} \bigoplus_{w \in W^{(i-1)}} \mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{w \cdot \lambda}) &\longrightarrow \bigoplus_{w' \in W^{(i)}} \mathcal{C}_{\mathrm{c}}^{\mathrm{lp}}(N, K_{w' \cdot \lambda}) \\ (f_w)_{w \in W^{(i-1)}} &\longmapsto \left(\sum_{\substack{\alpha \in \Phi^+ \\ l(s_\alpha w') = i-1}} \theta^{\mathrm{lp}}_{\alpha, s_\alpha w' \cdot \lambda}(f_{s_\alpha w'}) \right)_{w' \in W^{(i)}} \end{aligned}$$

Using the same formulae with $\theta^{\mathrm{lp}}_{\alpha, s_\alpha w' \cdot \lambda}$ replaced with $\theta^{\mathrm{la}}_{\alpha, s_\alpha w' \cdot \lambda}$ we get the following sequence of $\mathcal{U}(\mathfrak{g})$-modules:

$$\begin{aligned} 0 \to V \otimes_K \mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(N, K) \xrightarrow{d_{-1}} \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\lambda) \xrightarrow{d_0} \bigoplus_{w \in W^{(1)}} \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_{w \cdot \lambda}) \xrightarrow{d_1} \\ \cdots \xrightarrow{d_{i-1}} \bigoplus_{w \in W^{(i)}} \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_{w \cdot \lambda}) \xrightarrow{d_i} \cdots \xrightarrow{d_{r-1}} \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_{w_0 \cdot \lambda}) \to 0 \end{aligned} \tag{5.5}$$

Since (5.4) is exact and $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)$ is dense in $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)$, we know that $d_i \circ d_{i-1} = 0$ for all i and hence (5.5) is a chain complex. We will prove that it is in fact an exact sequence in Corollary 29.

Chapter 6

Exactness of the First Three Terms

Fix $\mu \in X(\mathbf{T})$ and choose an ordering $\alpha_1, \cdots, \alpha_r$ of Φ^+. We will now construct a basis for $\mathcal{C}^{\mathrm{pol}}(N, K_\mu)$ which diagonalises the action of $\mathfrak{h}$.

Theorem 17. *We can find $T_1, \cdots, T_r \in \mathcal{C}^{pol}(N, K_\mu)$ such that $\mathcal{C}^{pol}(N, K_\mu) \cong K[T_1, \cdots, T_r]$ and $T_1^{m_1} \cdots T_r^{m_r}$ is a weight vector of weight $\mu - \sum m_i\alpha_i$.*

Proof. Recall we have an isomorphism of $\mathcal{U}(\mathfrak{g})$-modules $\zeta_\mu : M_{\overline{\mathfrak{b}}}(\mu)^\vee \to \mathcal{C}^{\mathrm{pol}}(N, K_\mu)$. Write E_i for E_{α_i}. By the Poincaré-Birkhoff-Witt theorem $\{E_1^{n_1} \cdots E_r^{n_r} \otimes 1 : n_i \geq 0$ for all $i\}$ is a basis for $M_{\overline{\mathfrak{b}}}(\mu)$. Let $\{\varepsilon_{m_1,\cdots,m_r} : m_i \geq 0$ for all $i\}$ be the dual basis for $M_{\overline{\mathfrak{b}}}(\mu)^\vee$, defined by

$$\varepsilon_{m_1,\cdots,m_r}(E_1^{n_1} \cdots E_r^{n_r} \otimes 1) = \begin{cases} 1 & \text{if } m_i = n_i \text{ for all } i \\ 0 & \text{else.} \end{cases}$$

Since $E_1^{n_1} \cdots E_r^{n_r} \otimes 1$ has weight $-\mu + \sum n_i\alpha_i$, it follows that $\varepsilon_{m_1,\cdots,m_r}$ has weight $\mu - \sum m_i\alpha_i$. (Recall that $X \in \mathfrak{g}$ acts on $\phi \in M_{\overline{\mathfrak{b}}}(\mu)^\vee$ via $X\phi(u \otimes 1) = \phi(-Xu \otimes 1)$ for all $u \otimes 1 \in M_{\overline{\mathfrak{b}}}(\mu)$.)

We define $T_i \in \mathcal{C}^{\mathrm{pol}}(N, K_\mu)$ by $T_i = \zeta_\mu(\varepsilon_{0,\cdots,0,1,0,\cdots,0})$, where the 1 is in the ith place. Using Lemmas 18 and 19 which follow this proof we see that $\zeta_\mu(\varepsilon_{m_1,\cdots,m_r}) = T_1^{m_1} \cdots T_r^{m_r}/m_1! \cdots m_r!$, so $\mathcal{C}^{\mathrm{pol}}(N, K_\mu) = K[T_1, \cdots, T_r]$, and $T_1^{m_1} \cdots T_r^{m_r}$ has weight $\mu - \sum m_i\alpha_i$ since ζ_μ is $\mathcal{U}(\mathfrak{g})$-equivariant. $\square$

Remark. *If $\mu = w \cdot \lambda$ and $\alpha_r \in \Delta$ such that $l(s_{\alpha_r}w) = l(w) + 1$ then $\theta^{la}_{\alpha_r, w\cdot\lambda}$ is a non-zero scalar multiple of $(\frac{\partial}{\partial T_r})^{w\cdot\lambda(H_\alpha)+1}$.*

Here are the two lemmas about ζ_μ which were used in the proof.

Lemma 18. $\zeta_\mu(\varepsilon_{m_1,\cdots,m_r}) = \zeta_\mu(\varepsilon_{m_1,0,\cdots,0})\zeta_\mu(\varepsilon_{0,m_2,0,\cdots,0})\cdots\zeta_\mu(\varepsilon_{0,\cdots,0,m_r})$

Proof. The Leibniz rule says that for any $X, Y \in \mathcal{U}(\mathfrak{n})$ and $f, g \in \mathcal{C}^{\text{pol}}(N, K_\mu)$ we have $(XYfg)(e) = (XYf)(e)g(e) + (Xf)(e)(Yg)(e) + (Yf)(e)(Xg)(e) + f(e)(XYg)(e)$. It follows from this that $S(E_r^{n_r}\cdots E_1^{n_1})(\zeta_\mu(\varepsilon_{m_1,0,\cdots,0})\zeta_\mu(\varepsilon_{0,m_2,0,\cdots,0})\cdots\zeta_\mu(\varepsilon_{0,\cdots,0,m_r}))(e) = 0$ unless $m_i = n_i$ for all i, in which case it equals 1. This is the defining characteristic of $\zeta_\mu(\varepsilon_{m_1,\cdots,m_r})$. ☐

Lemma 19. *For all $m \geq 1$ we have $\zeta_\mu(\varepsilon_{0,\cdots,m,\cdots,0}) = \frac{1}{m!}\zeta_\mu(\varepsilon_{0,\cdots,1,\cdots,0})^m$, where all the indices are 0 except the ith.*

Proof. For $m = 1$ the result is trivial. Suppose it is true for $m-1$. As explained in Lemma 18 we have that $S(E_r^{n_r}\cdots E_1^{n_1})(\zeta_\mu(\varepsilon_{0,\cdots,1,\cdots,0})\zeta_\mu(\varepsilon_{0,\cdots,m-1,\cdots,0}))(e) = 0$ unless $n_i = m$ and $n_j = 0$ for all $j \neq i$, in which case it is

$$\binom{m}{1} S(E_i)(\zeta_\mu(\varepsilon_{0,\cdots,1,\cdots,0}))(e)\; S(E_i^{m-1})(\zeta_\mu(\varepsilon_{0,\cdots,m-1,\cdots,0}))(e)$$

which is m. Hence

$$\begin{aligned}\zeta_\mu(\varepsilon_{0,\cdots,m,\cdots,0}) &= \frac{1}{m}\zeta_\mu(\varepsilon_{0,\cdots,1,\cdots,0})\;\zeta_\mu(\varepsilon_{0,\cdots,m-1,\cdots,0})\\ &= \frac{1}{m}\zeta_\mu(\varepsilon_{0,\cdots,1,\cdots,0})\;\frac{1}{(m-1)!}\zeta_\mu(\varepsilon_{0,\cdots,1,\cdots,0})^{m-1}\\ &= \frac{1}{m!}\zeta_\mu(\varepsilon_{0,\cdots,1,\cdots,0})^m\end{aligned}$$

and by induction we are done. ☐

Let $\mathbb{B}$ denote the rigid analytic closed unit ball of dimension $r = \dim(N)$ defined over L. Let X be a compact, open subset of N such that there is a locally analytic isomorphism $X \cong \mathbb{B}(L)$ which is compatible with all charts of N. We write $\mathcal{C}^{\text{pol}}(X, K_\mu)$ for the subspace of $\mathcal{C}^{\text{la}}(X, K_\mu)$ given by restricting functions in $\mathcal{C}^{\text{pol}}(N, K_\mu)$ to X. Then $\mathcal{C}^{\text{pol}}(X, K_\mu) \subset \mathcal{C}^{\text{an}}(X, K_\mu)$ and we give it the norm coming from the Gauss norm on $\mathcal{C}^{\text{an}}(X, K_\mu)$.

Since $\mathfrak{g}$ acts on $\mathcal{C}^{\text{la}}_{\text{c}}(N, K_\mu)$ by differential operators, $\mathcal{C}^{\text{la}}(N, K_\mu)(X)$ is an $\mathcal{U}(\mathfrak{g})$-invariant subspace. Using the natural isomorphism we transfer this action of $\mathcal{U}(\mathfrak{g})$ to $\mathcal{C}^{\text{la}}(X, K_\mu)$, and to its $\mathcal{U}(\mathfrak{g})$-invariant subspaces $\mathcal{C}^{\mathrm{an}}(X, K_\mu)$ and $\mathcal{C}^{\mathrm{pol}}(X, K_\mu)$. This makes the map

$$\mathcal{C}^{\mathrm{pol}}(N, K_\mu) \longrightarrow \mathcal{C}^{\mathrm{pol}}(X, K_\mu) \qquad\qquad f \longmapsto f|_X$$

an isomorphism of $\mathcal{U}(\mathfrak{g})$-modules. So $\mathcal{C}^{\mathrm{pol}}(X, K_\mu)$ can be seen as a copy of $\mathcal{C}^{\mathrm{pol}}(N, K_\mu)$ with a norm coming from convergence on X.

We now use the basis we have just constructed for $\mathcal{C}^{\mathrm{pol}}(N, K_\mu)$ to study $\mathcal{C}^{\mathrm{pol}}(X, K_\mu)$.

The weights of $M_{\overline{\mathfrak{b}}}(\nu)$ are precisely $\{-\nu + \sum_{\delta\in\Delta} n_\delta \delta : n_\delta \geq 0\} \subset \mathfrak{h}^*$. Hence the weights of $M_{\overline{\mathfrak{b}}}(\nu)^\vee$ are $Z_\nu = \{\nu - \sum_{\delta\in\Delta} n_\delta\delta : n_\delta \geq 0\}$. Since $M_{\overline{\mathfrak{b}}}(\nu)^\vee \cong \mathcal{C}^{\mathrm{pol}}(N, K_\mu) \cong \mathcal{C}^{\mathrm{pol}}(X, K_\mu)$ as $\mathcal{U}(\mathfrak{g})$-modules, this is also the set of weights of $\mathcal{C}^{\mathrm{pol}}(X, K_\mu)$.

Lemma 20. *Suppose* $0 \in X$. *Then any* $f \in \mathcal{C}^{an}(X, K_\mu)$ *can be written uniquely as* $\sum_{\nu\in Z_\mu} f_\nu$ *where* $f_\nu \in \mathcal{C}^{pol}(X, K_\mu)_\nu$ *for each* $\nu \in Z_\mu$.

Proof. Choose $T_1, \cdots, T_r \in \mathcal{C}^{\mathrm{pol}}(N, K_\mu)$ as in Theorem 17. Replacing each T_i with its restriction to X we get $\mathcal{C}^{\mathrm{pol}}(X, K_\mu) = K[T_1, \cdots, T_r]$, so $Z_\mu = \{\mu - \sum n_i\alpha_i : n_i \geq 0$ for all $i\}$. Rescale them so that $|T_i| = 1$ for each i. Then $\mathcal{C}^{\mathrm{an}}(X, K_\mu)$ is the affinoid algebra

$$K\langle T_1, \cdots, T_r\rangle = \left\{\sum a_n T_1^{n_1}\cdots T_r^{n_r} : |a_n| \to 0 \text{ as } n_1 + \cdots + n_r \to \infty\right\}$$

with norm $\|\sum a_n T_1^{n_1}\cdots T_r^{n_r}\| = \sup|a_n|$. For $f = \sum_n a_n T_1^{n_1}\cdots T_r^{n_r} \in K\langle T_1, \cdots, T_r\rangle$ and $\nu \in Z_\mu$ we define $f_\nu = \displaystyle\sum_{n:\sum n_i\alpha_i = \mu-\nu} a_n T_1^{n_1}\cdots T_r^{n_r} \in \mathcal{C}^{\mathrm{pol}}(X, K_\mu)_\nu$. As required, $f = \sum_{\nu\in Z_\mu} f_\nu$.

Suppose $f = \sum_{\nu\in Z_\mu} f_\nu = \sum_{\nu\in Z_\mu} f'_\nu$, with f_ν and $f'_\nu \in \mathcal{C}^{\mathrm{pol}}(X, K_\mu)_\nu$. Then $0 = \sum_{\nu\in Z_\mu}(f_\nu - f'_\nu)$ and by considering coefficients of the $T_1^{n_1}\cdots T_r^{n_r}$ we see that $f_\nu = f'_\nu$ for all $\nu \in Z_\mu$. Thus the expression is unique. □

Corollary 21. *Suppose* $0 \in X$. *Then* $\mathcal{C}^{an}(X, K_\mu)_\nu = \mathcal{C}^{pol}(X, K_\mu)_\nu$ *for all* $\mu, \nu \in X(\mathbf{T})$.

Proof. Clearly $\mathcal{C}^{\mathrm{pol}}(X, K_\mu)_\nu \subseteq \mathcal{C}^{\mathrm{an}}(X, K_\mu)_\nu$. Let f be a non-zero element in $\mathcal{C}^{\mathrm{an}}(X, K_\mu)_\nu$. By Lemma 20 $f = \sum_{\eta\in Z_\mu} f_\eta$, so for all $Y \in \mathfrak{h}$ we have $0 = \nu(Y)f - Yf = \sum_{\eta\in Z_\mu}(\nu(Y) - \eta(Y))f_\eta$. Since the unique expression for 0 is $\sum 0$ we must have $(\nu(Y) - \eta(Y))f_\eta = 0$ for all $\eta \in Z_\mu$. Hence for all $\eta \neq \nu$ we have $f_\eta = 0$, and thus we must have $\nu \in Z_\mu$ and $f = f_\nu \in \mathcal{C}^{\mathrm{pol}}(X, K_\mu)_\nu$. □

Lemma 22. *We get an exact sequence*

$$0 \to V \xrightarrow{\delta_{-1}} \mathcal{C}^{an}(X, K_\lambda) \xrightarrow{\delta_0} \bigoplus_{w \in W^{(1)}} \mathcal{C}^{an}(X, K_{w\cdot\lambda})$$

by restricting the first three terms of (5.5).

Proof. We use d_i to refer to the maps in (5.5). Define δ_{-1} by $v \mapsto d_{-1}(v \otimes \mathbf{1}_X)|_X$. Let ϕ denote the map $V \to \mathcal{C}^{\mathrm{pol}}(N, K_\lambda)$ from (5.3), so $d_{-1}(\sum v_i \otimes \mathbf{1}_{X_i}) = \sum \phi(v_i)_{|X_i}$. It follows that $\delta_{-1}(v) = \phi(v)|_X$, from which it is easily seen that δ_{-1} is well-defined, injective and has $\operatorname{im}\delta_{-1} \subseteq \mathcal{C}^{\mathrm{pol}}(X, K_\lambda)$.

Given $f \in \mathcal{C}^{\mathrm{an}}(X, K_\mu)$ we can extend it by 0 to get $\overline{f} \in \mathcal{C}^{\mathrm{la}}_{\mathrm{c}}(N, K_\mu)$. We define δ_0 by sending $f \mapsto d_0(\overline{f})|_X$, where each component is restricted to X. As $d_0 = \bigoplus_{\alpha\in\Delta} \theta^{\mathrm{la}}_{\alpha,\lambda}$, this is well-defined by Lemma 15.4. For any $v \in V$

$$\delta_0(\delta_{-1}(v)) = d_0(d_{-1}(v \otimes \mathbf{1}_X)_{|X}) = d_0(d_{-1}(v \otimes \mathbf{1}_X))_{|X} = 0_{|X} = 0$$

using Lemma 15.1 and the fact that $d_0 \circ d_{-1} = 0$. Thus $\delta_0 \circ \delta_{-1} = 0$, and it only remains to show that $\ker\delta_0 \subseteq \operatorname{im}\delta_{-1}$.

Let $f \in \ker\delta_0$ and suppose we can show that $\ker\delta_0 \subseteq \mathcal{C}^{\mathrm{pol}}(X, K_\lambda)$. Then $\overline{f} \in \ker d_0$ is in $\mathcal{C}^{\mathrm{lp}}_{\mathrm{c}}(N, K_\lambda)$, so by exactness of (5.4) we can find $\sum v_i \otimes \mathbf{1}_{X_i} \in V \otimes \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(N, K)$ such that $d_{-1}(\sum v_i \otimes \mathbf{1}_{X_i}) = \overline{f}$. We may assume that the X_i are disjoint charts of N and that $X_i \subseteq X$ for all i. Let us compare $d_{-1}(\sum v_i \otimes \mathbf{1}_{X_i}) = \overline{f}$ with $d_{-1}(v_1 \otimes \mathbf{1}_X) = \phi(v_1)_{|X}$. They are both analytic on X and they agree on the non-empty open subset X_1, so they must agree on all of X. Hence $f = \phi(v_1)|_X = \delta_{-1}(v_1)$ and we have shown that $f \in \operatorname{im}\delta_{-1}$.

To complete the proof it suffices to show that $\ker\delta_0 \subseteq \mathcal{C}^{\mathrm{pol}}(X, K_\lambda)$. Fix $f \in \ker\delta_0$ and $n \in X$. Let $f' \in \mathcal{C}^{\mathrm{la}}_{\mathrm{c}}(Xn^{-1}, K_\lambda)$ denote $(n\overline{f})|_{Xn^{-1}}$. It is easy to see that in fact $f' \in \mathcal{C}^{\mathrm{an}}(Xn^{-1}, K_\lambda)$ and f' is in $\mathcal{C}^{\mathrm{pol}}(Xn^{-1}, K_\lambda)$ if and only if $f \in \mathcal{C}^{\mathrm{pol}}(X, K_\lambda)$. Thus it suffices to prove that $f' \in \mathcal{C}^{\mathrm{pol}}(Xn^{-1}, K_\lambda)$.

As $d_0 = \bigoplus \theta^{\mathrm{la}}_{\alpha,\lambda}$ where the sum is over all simple roots α, we have that $\theta^{\mathrm{la}}_{\alpha,\lambda}(\overline{f}) = 0$ for all $\alpha \in \Delta$. By Lemma 15.5 it follows that for all $\alpha \in \Delta$

$$\theta^{\mathrm{la}}_{\alpha,\lambda}(\overline{f'}) = \theta^{\mathrm{la}}_{\alpha,\lambda}(n\overline{f}) = n\theta^{\mathrm{la}}_{\alpha,\lambda}(\overline{f}) = 0$$

where $\overline{f'}$ means the extension of f' by 0 from Xn^{-1} to all of N.

By Lemma 20 we can write f' as $\sum_{\nu \in Z_\lambda} g_\nu$ with $g_\nu \in \mathcal{C}^{\text{pol}}(Xn^{-1}, K_\lambda)_\nu$. For any $\alpha \in \Delta$, $\theta^{\text{la}}_{\alpha,\lambda}$ preserves weights and $\theta^{\text{la}}_{\alpha,\lambda}(\sum_{\nu \in Z_\lambda} \overline{g_\nu}) = \sum_{\nu \in Z_\lambda} \theta^{\text{la}}_{\alpha,\lambda}(\overline{g_\nu})$, so by the uniqueness of Lemma 20 we must have $\theta^{\text{la}}_{\alpha,\lambda}(\overline{g_\nu}) = 0$ for each $\nu \in Z_\lambda$. This is true for all simple roots, so $\delta_0(g_\nu) = 0$. By the exactness of (5.4) we can therefore find $v_\nu \in V$ such that $\delta_{-1}(v_\nu) = g_\nu$. In fact, as δ_{-1} is $\mathcal{U}(\mathfrak{g})$-equivariant we must have $v_\nu \in V_\nu$. But since V is finite dimensional $V_\nu = 0$ for all but finitely many weights. Hence $g_\nu = 0$ for all but finitely many weights, and so $f' \in \bigoplus_{\nu \in Z_\lambda} \mathcal{C}^{\text{pol}}(Xn^{-1}, K_\lambda)_\nu = \mathcal{C}^{\text{pol}}(Xn^{-1}, K_\lambda)$. □

Theorem 23. *The first three terms of* (5.5)

$$0 \to V \otimes_K \mathcal{C}^{sm}_c(N, K) \xrightarrow{d_{-1}} \mathcal{C}^{la}_c(N, K_\lambda) \xrightarrow{d_0} \bigoplus_{w \in W^{(1)}} \mathcal{C}^{la}_c(N, K_{w \cdot \lambda})$$

form an exact sequence.

Proof. It follows from the exactness of (5.4) that d_{-1} is injective and $d_0 \circ d_{-1} = 0$. It only remains to prove that $\ker d_0 \subseteq \operatorname{im} d_{-1}$.

Let us fix $f \in \ker d_0$. By the definition of f being locally analytic with compact support, we can find a finite set of disjoint charts $\{X_i : i \in I\}$ of N such that $f|_{X_i}$ is analytic for each i and f is 0 outside $\bigcup X_i$. Applying Lemma 22 with $X = X_i$ we get $v_i \in V$ such that $f_{|X_i} = d_{-1}(v_i \otimes \mathbf{1}_{X_i})$. Hence $f = d_{-1}(\sum_{i \in I} v_i \otimes \mathbf{1}_{X_i})$ and we deduce that $\ker d_0 \subseteq \operatorname{im} d_{-1}$. □

Chapter 7

Locally analytic principal series for G_1 with an Iwahori factorisation

In this chapter we complete the proof that (5.5) is exact. To do this we have to introduce a particular kind of open compact subgroup of G.

Definition 24. *We say an open compact subgroup $G_1 \subseteq G$ **admits an Iwahori factorisation** (with respect to B and $\overline{B}$) if multiplication induces an isomorphism of L-analytic manifolds*

$$(\overline{N}_1) \times (T_1) \times (N_1) \xrightarrow{\sim} G_1$$

where $\overline{N}_1 = \overline{N} \cap G_1$, $T_1 = T \cap G_1$ and $N_1 = N \cap G_1$.

The canonical example of an open compact subgroup of G with an Iwahori factorisation is the Iwahori subgroup contained in a given special good maximal compact subgroup of G, and of type a given Borel subgroup. These are far from the only examples – indeed by Proposition 4.1.6 of [10] we can find arbitrarily small such subgroups. Let us fix an open compact subgroup $G_1 \subseteq G$ which admits an Iwahori factorisation.

Definition 25. *Let $\chi : T_1 \to GL_1(K)$ be a locally analytic character. The **locally analytic principal series** associated to G_1 and χ is $\mathrm{Ind}_{\overline{B}_1}^{G_1}(\chi)$.*

This has an action of G_1 by right translation. Since $(\overline{B}_1)\backslash G_1 \cong N_1$ is compact, it follows from 4.1.5 of [12] that this is a locally analytic representation of G_1, and so we can differentiate the G_1-action to get an action of $\mathcal{U}(\mathfrak{g})$. We can identify $\mathrm{Ind}_{B_1}^{G_1}(\chi)$ with $\mathrm{Ind}_{\overline{B}}^{G}(\chi)(N_1)$ using extension by 0, and hence with $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\chi)(N_1)$. Both of these maps are isomorphisms of $(\mathfrak{g}, B_1)$-modules, and we use them to transfer the action of G_1 to $\mathrm{Ind}_{\overline{B}}^{G}(\chi)(N_1)$ and $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\chi)(N_1)$.

Lemma 26. *The representation* $\mathrm{Ind}_{B_1}^{G_1}(\chi)$ *is an admissible representation of* G_1.

Proof. Proposition 6.4.iii of [20] says that a closed G_1-invariant subspace of an admissible G_1-representation is an admissible G_1-representation. It is thus sufficient to show that $\mathcal{C}^{\mathrm{la}}(G_1, K)$ is admissible. The topological dual of $\mathcal{C}^{\mathrm{la}}(G_1, K)$ is $D(G_1, K)$, which is coadmissible by Theorem 5.1 of [20] and the definition of a Fréchet-Stein algebra. □

In fact $\mathrm{Ind}_{B_1}^{G_1}(\chi)$ has an action of a monoid containing G_1. We define $T^- = \{t \in T : t^{-1}N_1 t \subseteq N_1\}$, which is a submonoid of T. We define M to be the submonoid of G generated by G_1 and T^-. Then $\overline{B}N_1M \subseteq \overline{B}N_1$, so the action of M on $\mathrm{Ind}_{\overline{B}}^{G}(\chi)$ preserves $\mathrm{Ind}_{\overline{B}}^{G}(\chi)(N_1)$. Using our earlier identifications we transfer this action of M to $\mathrm{Ind}_{B_1}^{G_1}(\chi)$ and $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\chi)(N_1)$.

Lemma 27. *Given a non-zero morphism* $\psi : M_{\overline{\mathfrak{b}}}(\mu_2) \to M_{\overline{\mathfrak{b}}}(\mu_1)$*, let* $\varphi : \mathcal{C}_c^{la}(N, K_{\mu_1})(N_1) \to \mathcal{C}_c^{la}(N, K_{\mu_2})(N_1)$ *denote the function obtained by restricting* ψ^{la}*. Then* φ *is* M*-equivariant.*

Proof. That φ is well-defined follows from Lemma 15.2. Using the M-equivariant isomorphism $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N, K_\mu)(N_1) \to \mathrm{Ind}_{\overline{B}}^{G}(\mu)(N_1)$ we can turn φ into a map $\mathrm{Ind}_{\overline{B}}^{G}(\mu_1)(N_1) \to \mathrm{Ind}_{\overline{B}}^{G}(\mu_2)(N_1)$. This map is precisely $(u_\psi)_L$, and the L action of $\mathfrak{g}$ commutes with the right regular action of M. □

We define the smooth induction of the trivial character

$$\text{sm-Ind}_{\overline{B}}^{G}(\mathbf{1}) = \{f \in \mathcal{C}^{\mathrm{sm}}(G, K) : f(\overline{b}g) = f(g) \text{ for all } \overline{b} \in \overline{B}, g \in G\}$$

and we have $\mathcal{C}^{\mathrm{sm}}(N_1, K) \cong \text{sm-Ind}_{B_1}^{G_1}(\mathbf{1}) \cong \text{sm-Ind}_{\overline{B}}^{G}(\mathbf{1})(N_1)$ as $\mathcal{U}(\mathfrak{g})$-modules. The same argument as for $\mathrm{Ind}_{B_1}^{G_1}(\chi)$ gives us an action of M on all of these spaces.

Recall that $Z_\nu = \{\nu - \sum_{\delta \in \Delta} n_\delta \delta : n_\delta \geq 0\} \subset \mathfrak{h}^*$. Since λ is dominant, for any $w \in W$ we can write $\lambda - w \cdot \lambda$ as $\sum_{\delta \in \Delta} m_\delta \delta$ with all the $m_\delta \geq 0$. Therefore $Z_{w \cdot \lambda} \subset Z_\lambda$.

Proposition 28. *When we restrict* (5.5) *to functions with support in* N_1 *we get an exact sequence of* M*-representations*

$$0 \to V \otimes \mathcal{C}_c^{sm}(N,K)(N_1) \xrightarrow{\delta_{-1}} \mathcal{C}_c^{la}(N,K_\lambda)(N_1) \xrightarrow{\delta_0} \bigoplus_{w \in W^{(1)}} \mathcal{C}_c^{la}(N,K_{w\cdot\lambda})(N_1)$$

$$\cdots \xrightarrow{\delta_{i-1}} \bigoplus_{w \in W^{(i)}} \mathcal{C}_c^{la}(N,K_{w\cdot\lambda})(N_1) \xrightarrow{\delta_i} \cdots \xrightarrow{\delta_{r-1}} \mathcal{C}_c^{la}(N,K_{w_0\cdot\lambda})(N_1) \to 0$$

Proof. For $i \geq 0$, δ_i is well-defined by Lemma 15.2 and M-equivariant by Lemma 27. That δ_{-1} is well-defined and M-equivariant follows immediately from the definition of d_{-1}. (We use d_i to refer to the maps in (5.5).)

From Theorem 23 we see that δ_{-1} is an injection and $\delta_0 \circ \delta_{-1} = 0$. We can also deduce $\ker \delta_0 \subseteq \operatorname{im} \delta_{-1}$, by the following argument. Suppose that $f \in \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N,K_\lambda)(N_1)$ is in $\ker \delta_0$. Then we know that it has a preimage $\sum v_i \otimes \mathbf{1}_{X_i} \in V \otimes_K \mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(N,K)$ where the X_i are disjoint charts of N. If we let ϕ denote the injection $V \to \mathcal{C}^{\mathrm{pol}}(N,K_\lambda)$ from (5.3) then $d_{-1}(\sum v_i \otimes \mathbf{1}_{X_i}) = \sum \phi(v_i)\mathbf{1}_{X_i}$, whence it follows that $X_i \subseteq N_1$ for all i and $f \in \operatorname{im} \delta_{-1}$.

We now prove exactness at $\bigoplus_{w\in W^{(i)}} \mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N,K_{w\cdot\lambda})(N_1)$ for $i \geq 1$. Since $d_i \circ d_{i-1} = 0$ we know that $\delta_i \circ \delta_{i-1} = 0$, so it suffices to prove that $\ker \delta_i \subseteq \operatorname{im} \delta_{i-1}$.

Fix $(f_w)_{w\in W^{(i)}} \in \ker \delta_i$. Let us first suppose that we have a chart $X \subseteq N_1$ such that each f_w is analytic on X and 0 outside it, and let us further suppose that $0 \in X$. Since $Z_\lambda \supseteq Z_{w\cdot\lambda}$ for all $w \in W$, by Lemma 20, we can write each f_w uniquely as $\sum_{\nu\in Z_\lambda} f_{w,\nu}$ where $f_{w,\nu}|_X \in \mathcal{C}^{\mathrm{pol}}(X,K_{w\cdot\lambda})_\nu$ and $f_{w,\nu}$ is 0 outside X. Using the fact that δ_i is $\mathcal{U}(\mathfrak{g})$-equivariant, and applying Lemma 20 with $\mu = w\cdot\lambda$ for each $w \in W^{(i+1)}$, we see that $(f_{w,\nu})_{w\in W^{(i)}} \in \ker \delta_i$ for each $\nu \in Z_\lambda$. Since Z_λ is countable let us choose an increasing sequence of finite subsets $A_n \subseteq Z_\lambda$ such that $\bigcup_{n=1}^\infty A_n = Z_\lambda$ and set $f_{w,n} = \sum_{\nu\in A_n} f_{w,\nu}$. Then $(f_{w,n})_{w\in W^{(i)}}$ tends to $(f_w)_{w\in W^{(i)}}$ as $n \to \infty$. By the exactness of (5.4), each $(f_{w,n})_{w\in W^{(i)}}$ is in $\operatorname{im} d_{i-1}$, and hence in $\operatorname{im} \delta_{i-1}$ by Lemma 15.3. We want to show that their limit must therefore also be in $\operatorname{im} \delta_{i-1}$. It is sufficient to demonstrate that $\operatorname{im} \delta_{i-1}$ is closed.

As explained in Lemma 26, $\mathcal{C}_{\mathrm{c}}^{\mathrm{la}}(N,K_\nu)(N_1) \cong \operatorname{Ind}_{B_1}^{G_1}(\nu)$ is an admissible G_1-representation, and hence δ_{i-1} is a G_1-equivariant, K-linear map between two admissible G_1-representations. By Proposition 6.4.ii in [20], the image of δ_{i-1} is closed.

Now suppose that $0 \notin X$. Choose $n \in X$. Replacing each f_w with nf_w and using the chart Xn^{-1}, by the above argument we have that $(nf_w)_{w\in W^{(i)}} \in \operatorname{im}\delta_{i-1}$, say $(nf_w)_{w\in W^{(i)}} = \delta_{i-1}((g_w)_{w\in W^{(i-1)}})$. Then $\delta_{i-1}((n^{-1}g_w)_{w\in W^{(i-1)}}) = (f_w)_{w\in W^{(i)}}$ and hence $(f_w)_{w\in W^{(i)}} \in \operatorname{im}\delta_{i-1}$.

For a general $(f_w)_{w\in W^{(i)}} \in \ker\delta_i$ we can find a finite set of disjoint charts $\{X_j\}$ which cover N_1 and such that for all $w \in W^{(i)}$ and all j, f_w is analytic on X_j. We know that $((f_w)_{|X_j})_{w\in W^{(i)}}$ is still in $\ker\delta_i$, so by the above arguments we can find a preimage for it, and adding these all together we get a preimage for $(f_w)_{w\in W^{(i)}}$. □

Corollary 29. *The sequence* (5.5) *is exact.*

Proof. Theorem 23 deals with exactness at $V\otimes_K \mathcal{C}_c^{\mathrm{sm}}(N,K)$ and $\mathcal{C}_c^{\mathrm{la}}(N,K_\lambda)$. Let $i\geq 1$. We know that $d_i\circ d_{i-1}=0$, so it only remains to show that $\ker d_i \subseteq \operatorname{im} d_{i-1}$.

First consider $(f_w)_{w\in W^{(i)}}$ in $\ker d_i$ such that for some $n\in N$, $\operatorname{Supp} f_w \subseteq N_1 n$ for all $w\in W^{(i)}$. We showed in Theorem 14 that ψ^{la} is B-equivariant, so d_i is too. Since $\operatorname{Supp} nf_w \subseteq N_1$ for all $w\in W^{(i)}$ we have that $d_i((nf_w)_{w\in W^{(i)}}) = nd_i((f_w)_{w\in W^{(i)}}) = 0$. We proved in Proposition 28 that we therefore have a preimage $(g_w)_{w\in W^{(i-1)}}$ of $(nf_w)_{w\in W^{(i)}}$. Then $((n^{-1}g_w)_{w\in W^{(i-1)}}$ is a preimage of $(f_w)_{w\in W^{(i)}}$.

A general $(f_w)_{w\in W^{(i)}} \in \ker d_i$ can be written as a finite sum of such functions, so by linearity we are done. □

Theorem 30. *We have an exact sequence of M-representations*

$$0\to V\otimes_K \text{sm-Ind}_{B_1}^{G_1}(\mathbf{1}) \to \mathrm{Ind}_{B_1}^{G_1}(\lambda) \to \bigoplus_{w\in W^{(1)}} \mathrm{Ind}_{B_1}^{G_1}(w\cdot\lambda) \to \cdots \to \bigoplus_{w\in W^{(i)}} \mathrm{Ind}_{B_1}^{G_1}(w\cdot\lambda) \to \cdots \to \mathrm{Ind}_{B_1}^{G_1}(w_0\cdot\lambda)\to 0$$

coming from the BGG resolution for V^*.

Proof. This follows immediately from Proposition 28. □

7.1 Analytic principal series for G_1 with an Iwahori factorisation

Let G_1 be an open compact subgroup of G which admits an Iwahori factorisation, and such that there is a locally analytic isomorphism $N_1 \cong \mathbb{B}(L)$ which is compatible with all charts of N. (Recall $\mathbb{B}$ is the rigid analytic closed unit ball of dimension r defined over L.)

Definition 31. *The **analytic principal series** associated to G_1 and $\mu \in X(\mathbf{T})$ is*

$$\text{an-Ind}_{\overline{B}_1}^{G_1}(\mu) = \{f \in \text{Ind}_{\overline{B}_1}^{G_1}(\mu) : f \textit{ is analytic on } N_1\}.$$

The action of $\mathcal{U}(\mathfrak{g})$ on $\text{Ind}_{\overline{B}_1}^{G_1}(\mu)$ preserves $\text{an-Ind}_{\overline{B}_1}^{G_1}(\mu)$ because the right regular action of $\mathfrak{g}$ on $\mathcal{C}^{\text{la}}(G, K)$ is via differential operators, which preserve the property of being analytic on N_1. We use this to give $\text{an-Ind}_{\overline{B}_1}^{G_1}(\mu)$ an action of $\mathcal{U}(\mathfrak{g})$.

Lemma 32. *The action of M on $\text{Ind}_{\overline{B}_1}^{G_1}(\mu)$ preserves $\text{an-Ind}_{\overline{B}_1}^{G_1}(\mu)$.*

Proof. Consider the image of $\text{an-Ind}_{\overline{B}_1}^{G_1}(\mu)$ under $\text{Ind}_{\overline{B}_1}^{G_1}(\mu) \cong \text{Ind}_{\overline{B}}^{G}(\mu)(N_1)$. Since μ is analytic it consists of all functions which are analytic on $\overline{B}N_1$ and 0 outside it. Since $\overline{B}N_1M = \overline{B}N_1$, this is preserved by the action of M. $\square$

We use this to give $\text{an-Ind}_{\overline{B}_1}^{G_1}(\mu)$ an action of M.

Theorem 33. *The sequence* (5.5) *gives an exact sequence of M-representations*

$$0 \longrightarrow V \longrightarrow \text{an-Ind}_{\overline{B}_1}^{G_1}(\lambda) \longrightarrow \bigoplus_{w \in W^{(1)}} \text{an-Ind}_{\overline{B}_1}^{G_1}(w \cdot \lambda).$$

Proof. Setting $X = N_1$ and using the isomorphism $\text{an-Ind}_{\overline{B}_1}^{G_1}(\mu) \cong \mathcal{C}^{\text{an}}(N_1, K_\mu)$, we showed this sequence was exact in Lemma 22. The maps are M-equivariant because they are the restriction of maps from the exact sequence in Theorem 30, which are M-equivariant, and we have shown the spaces are M-stable. $\square$

The analogue of the whole of (5.5) with analytic principal series is a chain complex but is not in general an exact sequence. Consider, for example, $G = GL_2(\mathbb{Q}_p)$. If $\Phi^+ = \{\alpha\}$ and $\lambda = n\alpha$ with $n \geq 0$ then the sequence we get is

$$0 \to V \xrightarrow{\delta_{-1}} \mathbb{Q}_p\langle T\rangle \xrightarrow{\delta_0} \mathbb{Q}_p\langle T\rangle \to 0$$

where the image of δ_{-1} is the space of polynomials of degree $\leq n$ and $\delta_0 = \left(\frac{\partial}{\partial T}\right)^{n+1}$. This sequence is not exact as δ_0 is not surjective: for example, $\sum p^i T^{p^{2i}}$ is not in its image.

Chapter 8

Locally analytic principal series for G

Using the results of Chapter 7 we now prove an analogue of Theorem 30 for locally analytic principal series for G. Let G_0 be a special good maximal compact subgroup of G. The Iwasawa decomposition says that $G = \overline{B}G_0$ (cf. §3.5 in [4]), which gives us an isomorphism of G_0-representations

$$\mathrm{Ind}_{\overline{B}}^{G}(\mu) \cong \mathrm{Ind}_{\overline{B}_0}^{G_0}(\mu)$$

where $\overline{B}_0 = \overline{B} \cap G_0$.

We may fix representatives of W which are in G_0, by 4.2.3 of [3].

Let $G_1 \subseteq G_0$ be the Iwahori subgroup of the same type as $\overline{B}$. This has an Iwahori factorisation with respect to B and $\overline{B}$ and we have the Bruhat-Iwahori decomposition

$$G_0 = \bigsqcup_{w \in W} \overline{B}_0 w G_1.$$

Hence any $f \in \mathrm{Ind}_{\overline{B}_0}^{G_0}(\mu)$ is determined by knowing $f|_{wG_1}$ for all $w \in W$, or equivalently by $(wf)|_{wG_1w^{-1}}$ for all $w \in W$. This gives us an isomorphism of G_1-representations

$$\mathrm{Ind}_{\overline{B}_0}^{G_0}(\mu) \longrightarrow \bigoplus_{w \in W} \mathrm{Ind}_{\overline{B} \cap wG_1w^{-1}}^{wG_1w^{-1}}(\mu) \qquad\qquad f \longmapsto ((wf)|_{wG_1w^{-1}})_{w \in W}$$

where the action of G_1 on $\mathrm{Ind}_{\overline{B} \cap wG_1w^{-1}}^{wG_1w^{-1}}(\mu)$ is via $G_1 \to wG_1w^{-1}$, $g \mapsto wgw^{-1}$.

Lemma 34. *For any $w \in W$, wG_1w^{-1} has an Iwahori factorisation*

$$(wG_1w^{-1} \cap \overline{N}) \times (wG_1w^{-1} \cap T) \times (wG_1w^{-1} \cap N) \xrightarrow{\sim} wG_1w^{-1}$$

with respect to B and $\overline{B}$.

Proof. This follows from Lemme 5.4.2 in [15]. $\square$

In Chapter 4 we started with a $\mathcal{U}(\mathfrak{g})$-equivariant map $\psi : M_{\overline{\mathfrak{b}}}(\mu_2) \to M_{\overline{\mathfrak{b}}}(\mu_1)$ and constructed a $(\mathfrak{g}, B)$-equivariant map $\psi^{\text{la}} : \mathcal{C}^{\text{la}}_{\text{c}}(N, K_{\mu_1}) \to \mathcal{C}^{\text{la}}_{\text{c}}(N, K_{\mu_2})$. In Lemma 27 we showed that ψ^{la} gives the G_1-equivariant map

$$(u_\psi)_L : \text{Ind}^{wG_1w^{-1}}_{\overline{B}\cap wG_1w^{-1}}(\mu_1) \to \text{Ind}^{wG_1w^{-1}}_{\overline{B}\cap wG_1w^{-1}}(\mu_2)$$

Lemma 35. *Using $\text{Ind}^G_{\overline{B}}(\mu) \cong \text{Ind}^{G_0}_{\overline{B}_0}(\mu) \cong \bigoplus_{w\in W} \text{Ind}^{wG_1w^{-1}}_{\overline{B}\cap wG_1w^{-1}}(\mu)$, the above maps give us a G_1-equivariant map $\text{Ind}^G_{\overline{B}}(\mu_1) \to \text{Ind}^G_{\overline{B}}(\mu_2)$. It is moreover G-equivariant.*

Proof. We will show that this map is precisely $(u_\psi)_L$, which is G-equivariant. Let $f \in \text{Ind}^G_{\overline{B}}(\mu_1)$. This corresponds to

$$((wf)|_{wG_1w^{-1}})_{w\in W} \in \bigoplus_{w\in W} \text{Ind}^{wG_1w^{-1}}_{\overline{B}\cap wG_1w^{-1}}(\mu_1)$$

which is in turn sent to

$$((u_\psi)_L(wf)|_{wG_1w^{-1}})_{w\in W} \in \bigoplus_{w\in W} \text{Ind}^{wG_1w^{-1}}_{\overline{B}\cap wG_1w^{-1}}(\mu_2).$$

There is a unique $f' \in \text{Ind}^G_{\overline{B}}(\mu_2)$ such that $(wf')|_{wG_1w^{-1}} = (u_\psi)_L(wf)|_{wG_1w^{-1}}$ for all $w \in W$. Since

$$(u_\psi)_L(wf)|_{wG_1w^{-1}} = ((u_\psi)_L wf)|_{wG_1w^{-1}} = (w(u_\psi)_L f)|_{wG_1w^{-1}}$$

the obvious candidate for f' is $(u_\psi)_L f$. We must show that $(u_\psi)_L f \in \text{Ind}^G_{\overline{B}}(\mu_2)$. Since $G = \overline{B}G_0 = \overline{B}\left(\bigsqcup_{w\in W} \overline{B}_0 wG_1\right) = \bigsqcup_{w\in W} \overline{B}wG_1$, it suffices to prove that $(u_\psi)_L f(\overline{b}wg) = \mu_2(\overline{b})(u_\psi)_L f(wg)$ for all $\overline{b} \in \overline{B}$, $w \in W$ and $g \in G_1$.

Fix $w \in W$. We have $wf \in \mathrm{Ind}_{\overline{B}}^{G}(\mu_1)$ and hence $(wf)_{|N} \in \mathrm{Ind}_{\overline{B}}^{G}(\mu_1)(N)$. In the proof of Theorem 14 we showed that $(u_\psi)_L \mathrm{Ind}_{\overline{B}}^{G}(\mu_1)(N) \subseteq \mathrm{Ind}_{\overline{B}}^{G}(\mu_2)(N)$, so $(u_\psi)_L((wf)_{|N}) \in \mathrm{Ind}_{\overline{B}}^{G}(\mu_2)(N)$. Let $\bar{b} \in \overline{B}$ and $g \in G_1$. By Lemma 11, $(u_\psi)_L((wf)_{|N}) = ((u_\psi)_L wf)_{|N}$, and $wgw^{-1} \in \overline{B}N$ by Lemma 34, so

$$\begin{aligned}((u_\psi)_L wf))(\bar{b}wgw^{-1}) &= (((u_\psi)_L wf)_{|N})(\bar{b}wgw^{-1}) \\ &= \mu_2(\bar{b})((u_\psi)_L wf)_{|N})(wgw^{-1}) \\ &= \mu_2(\bar{b})((u_\psi)_L wf))(wgw^{-1})\end{aligned}$$

whence it immediately follows that $(u_\psi)_L f(\bar{b}wg) = \mu_2(\bar{b})(u_\psi)_L f(wg)$. □

We can now prove an analogue of Theorem 30 for locally analytic principal series for all of G. This has been done independently by different methods in §4.9 of [16].

Theorem 36. *We have an exact sequence of G-representations*

$$\begin{aligned}0 \to V \otimes \text{sm-}\mathrm{Ind}_{\overline{B}}^{G}(\mathbf{1}) \to \mathrm{Ind}_{\overline{B}}^{G}(\lambda) \to \bigoplus_{w \in W^{(1)}} \mathrm{Ind}_{\overline{B}}^{G}(w \cdot \lambda) \to \\ \cdots \to \bigoplus_{w \in W^{(i)}} \mathrm{Ind}_{\overline{B}}^{G}(w \cdot \lambda) \to \cdots \to \mathrm{Ind}_{\overline{B}}^{G}(w_0 \cdot \lambda) \to 0\end{aligned}$$

coming from the BGG resolution for V^*.

Proof. For each $w \in W$ we have an exact sequence of wG_1w^{-1}-representations

$$\begin{aligned}0 \to V \otimes \text{sm-}\mathrm{Ind}_{\overline{B} \cap wG_1w^{-1}}^{wG_1w^{-1}}(\mathbf{1}) \to \mathrm{Ind}_{\overline{B} \cap wG_1w^{-1}}^{wG_1w^{-1}}(\lambda) \to \bigoplus_{w \in W^{(1)}} \mathrm{Ind}_{\overline{B} \cap wG_1w^{-1}}^{wG_1w^{-1}}(w \cdot \lambda) \\ \cdots \to \bigoplus_{w \in W^{(i)}} \mathrm{Ind}_{\overline{B} \cap wG_1w^{-1}}^{wG_1w^{-1}}(w \cdot \lambda) \to \cdots \to \mathrm{Ind}_{\overline{B} \cap wG_1w^{-1}}^{wG_1w^{-1}}(w_0 \cdot \lambda) \to 0\end{aligned}$$

by Lemma 34 and the results of Chapter 7. We turn the wG_1w^{-1}-action in an action of G_1 via $G_1 \to wG_1w^{-1}$. Taking the direct sum of all of these sequences and using the G_1-equivariant isomorphism $\mathrm{Ind}_{\overline{B}}^{G}(\mu) \cong \bigoplus_{w \in W} \mathrm{Ind}_{\overline{B} \cap wG_1w^{-1}}^{wG_1w^{-1}}(\mu)$ and its smooth analogue we get the required exact sequence, but only as an exact sequence of G_1-representations. It remains to show that the maps are G-equivariant.

First consider $d_{-1} : V \otimes \text{sm-Ind}_{\overline{B}}^{G}(\mathbf{1}) \to \text{Ind}_{\overline{B}}^{G}(\lambda)$. Given $v \otimes f \in V \otimes \text{sm-Ind}_{\overline{B}}^{G}(\mathbf{1})$ we construct $d_{-1}(v \otimes f)$ as follows. First we send $v \otimes f$ to

$$(wv \otimes wf|_{wG_1w^{-1}})_{w\in W} \in \bigoplus_{w\in W} V \otimes \text{sm-Ind}_{\overline{B}\cap wG_1w^{-1}}^{wG_1w^{-1}}(\mathbf{1}).$$

We then apply the maps

$$V \otimes \text{sm-Ind}_{\overline{B}\cap wG_1w^{-1}}^{wG_1w^{-1}}(\mathbf{1}) \to \text{Ind}_{\overline{B}\cap wG_1w^{-1}}^{wG_1w^{-1}}(\lambda) \qquad v \otimes f \mapsto \phi(v)|_{wG_1w^{-1}} f$$

where ϕ is the G-equivariant isomorphism from V to the algebraic induction of λ from $\overline{B}$ to G. This gives us $(\phi(wv)(wf)|_{wG_1w^{-1}})_{w\in W}$, which can be expressed as $(w(\phi(v)f)|_{wG_1w^{-1}})_{w\in W}$. Applying the inverse of the isomorphism $\text{Ind}_{\overline{B}_0}^{G_0}(\lambda) \cong \bigoplus_{w\in W} \text{Ind}_{\overline{B}\cap wG_1w^{-1}}^{wG_1w^{-1}}(\lambda)$ we get that $d_{-1}(v \otimes f) = \phi(v)f$ and hence d_{-1} is G-equivariant.

The G-equivariance of $d_i : \bigoplus_{w\in W^{(i)}} \text{Ind}_{\overline{B}}^{G}(w\cdot\lambda) \to \bigoplus_{w\in W^{(i+1)}} \text{Ind}_{\overline{B}}^{G}(w\cdot\lambda)$ for $i \geq 0$ follows easily from Lemma 35 and the fact that the maps $\bigoplus_{w\in W^{(i)}} \text{Ind}_{\overline{B}\cap wG_1w^{-1}}^{wG_1w^{-1}}(w\cdot\lambda) \to \bigoplus_{w\in W^{(i+1)}} \text{Ind}_{\overline{B}\cap wG_1w^{-1}}^{wG_1w^{-1}}(w\cdot\lambda)$ are constructed from maps of the form $(u_{\theta_{\alpha,w\cdot\lambda}})_L$. $\square$

The analogous result in [16] holds for induction from any parabolic subgroup, not just a Borel. It seems likely that our methods could also be adapted to treat this situation.

Chapter 9

Applications to overconvergent p-adic automorphic forms I

In this chapter we outline the definition of spaces of overconvergent p-adic automorphic forms given in [5] and construct an exact sequence between certain such spaces. This has already been done in [5] but is included here for completeness.

Let F be a number field. Let $\mathbf{U}$ be an algebraic group defined over F such that $\mathbf{U}(F_v)$ is compact for all infinite places v of F and $\mathbf{U}(F_v) \cong GL_n(\mathbb{Q}_p)$ for all places v of F dividing p. Let S_p denote the set of all places of F dividing p and fix an isomorphism $\mathbf{U}(F_v) \cong GL_n(\mathbb{Q}_p)$ for all $v \in S_p$.

Let $\mathbf{G}$ be the algebraic group $\mathbf{GL}_n^{S_p}$ defined over $\mathbb{Q}_p$. Let $G = \mathbf{G}(\mathbb{Q}_p)$, $B \subseteq G$ the Borel consisting of lower triangular matrices and $T \subseteq G$ the maximal torus consisting of diagonal matrices. Define $G_1 \subseteq G$ to be the Iwahori subgroup of $GL_n(\mathbb{Z}_p)^{S_p}$ coming from $\overline{B}$. (Because of differing conventions what we call $\mathbf{B}$ is called $\overline{\mathbf{B}}$ in [5] and vice versa.)

Let $\mathbb{A}_f$ denote the finite adèles over F and $\mathbb{A}_f^{S_p}$ the finite adèles over F away from $v \in S_p$. Fix an open compact subgroup $\mathscr{U}$ of $\mathbf{U}(\mathbb{A}_f)$ of the form $G_1 \times \mathscr{U}^{S_p}$ where $\mathscr{U}^{S_p}$ is an open compact subgroup of $\mathbf{U}(\mathbb{A}_f^{S_p})$. Let M be the submonoid of G generated by G_1 and $T^- = \{t \in T : t^{-1}N_1 t \subseteq N_1\}$. Consider the functor $\mathscr{F}$ from representations of M over $\mathbb{Q}_p$ to $\mathbb{Q}_p$-vector spaces given by setting $\mathscr{F}(A)$ to be the set of all functions $\phi : \mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A}_f) \to A$ such that $\phi(gx) = (\prod_{v|p} x_v)^{-1}\phi(g)$ for all $g \in \mathbf{U}(\mathbb{A}_f)$ and $x \in \mathscr{U}$. This is an exact functor.

For $\mu \in X(\mathbf{T})$ Chenevier defines a representation $\mathcal{C}_\mu$ of M which can easily be shown to be isomorphic to $\text{an-Ind}_{B_1}^{G_1}(-\mu)$. (Recall the group operation on $X(\mathbf{T})$ is written additively, so $(-\mu)(t) = \mu(t)^{-1}$.) He defines the space of automorphic forms of $\mathbf{U}$ of weight μ and level $\mathscr{U}$ to be $\mathscr{F}(\mathcal{C}_\mu)$.

Theorem 37. *Let V be a finite dimensional irreducible algebraic representation of $\mathbf{G}$, with lowest weight $\lambda \in X(\mathbf{T})$. We have an exact sequence*

$$0 \to \mathscr{F}(V^*) \to \mathscr{F}(\mathcal{C}_\lambda) \to \bigoplus_{w \in W^{(1)}} \mathscr{F}(\mathcal{C}_{w \cdot \lambda})$$

Proof. Consider the exact sequence in Theorem 33 with V replaced by V^*, which has highest weight $-\lambda$. Applying the functor $\mathscr{F}$ we get the required exact sequence. □

Note that when we talk about highest and lowest weights we mean with respect to the choice of positive roots given by $\mathbf{B}$. Since Chenevier takes our $\overline{\mathbf{B}}$ for his choice of positive roots, in his terminology V has highest weight λ.

Chenevier calls $\mathscr{F}(V^*)$ the space of classical overconvergent p-adic automorphic forms.

Chapter 10

Applications to overconvergent p-adic automorphic forms II

In this chapter we outline the definition of spaces of overconvergent p-adic automorphic forms given in [14] and construct an exact sequence involving them.

Choose a number field F and a prime $\mathfrak{p}$ of F. Let $\mathbf{H}$ be a connected reductive algebraic group defined over F such that $\mathbf{H}(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact modulo centre. Write H^0_∞ for the identity component of $\mathbf{H}(F \otimes_{\mathbb{Q}} \mathbb{R})$. Let $\mathbb{A}$ denote the adèles of F, $\mathbb{A}_f$ the finite adèles of F and $\mathbb{A}_f^{(\mathfrak{p})}$ the finite adèles of F away from $\mathfrak{p}$. Let $L = F_\mathfrak{p}$ and let $\mathbf{G}$ be the base change of $\mathbf{H}$ to L. Assume that $\mathbf{G}$ is quasi-split.

We are now in the situation of [14], with the added assumption that the parabolic subgroup $P \subseteq \mathbf{H}(F_\mathfrak{p})$ is a Borel. Let us now outline the definition of the space of overconvergent p-adic automorphic forms for $\mathbf{H}$ used in [14]. In the terminology of [14], we consider only the case where X in arithmetic weight space is in fact the singleton $\mathbf{1}$ consisting of the trivial weight and V is a one-dimensional representation of T_1 of the form K_μ for $\mu \in X(\mathbf{T})$ which is an arithmetical character. The field called E in [14] we call K, the group called G_0 we call G_1 and the monoid called $\mathbb{I}$ we call M. We put the extra condition on G_1 that if $t \in T$ such that $|\alpha(t)| < 1$ for all $\alpha \in \Delta$ then $tN_1t^{-1} \subseteq N_1$ and $t^{-1}\overline{N}_1 t \subseteq \overline{N}_1$.

Let M be the submonoid of G generated by G_1 and $T^- = \{t \in T : t^{-1}N_1 t \subseteq N_1\}$. A representation of M over K or a weight $\mu \in X(\mathbf{T})$ is said to be arithmetical if there is a finite index subgroup in $\mathbf{Z}_{\mathbf{H}}(\mathfrak{o}_F)$ which acts trivially.

For an arithmetical representation A of M over K we define $\mathcal{L}(A)$ to be the set of all functions $\phi : \mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A}) \to A$ such that there exists some open subset $U \subseteq \mathbf{H}(\mathbb{A}_f^{(\mathfrak{p})}) \times G_1$ (which can depend on ϕ) with $\phi(hu) = u_{\mathfrak{p}}^{-1}\phi(h)$ for all $u \in U \times H_\infty^0$ and $h \in \mathbf{H}(\mathbb{A})$.

For a sufficiently large integer k the space of k-overconvergent p-adic automorphic forms $M(e, \mathbf{1}, V, k)$ for $\mathbf{H}$ with weight $(\mathbf{1}, K_\mu)$ and type e is defined to be $e\mathcal{L}(\mathcal{C}(\mathbf{1}, K_\mu, k))$. Here e is an idempotent in a certain Hecke algebra $\mathcal{H}^+(\mathcal{G})$ which corresponds to the tame level – see [14] for more details, and for the definition of $\mathcal{C}(\mathbf{1}, K_\mu, k)$.

For k large enough that $\mathcal{C}(\mathbf{1}, K_\mu, k)$ is defined there is a natural map $\mathcal{C}(\mathbf{1}, K_\mu, k) \to \mathcal{C}(\mathbf{1}, K_\mu, k+1)$, so functoriality gives a map $e\mathcal{L}(\mathcal{C}(\mathbf{1}, K_\mu, k)) \to e\mathcal{L}(\mathcal{C}(\mathbf{1}, K_\mu, k+1))$ (which is injective with dense image). We make the following definition.

Definition 38. *The* ***space*** $M(e, K_\mu)$ ***of overconvergent*** p***-adic automorphic forms*** *of weight* K_μ *and type* e *is defined to be* $\varinjlim_k M(e, \mathbf{1}, K_\mu, k))$.

In the proof of Proposition 3.10.1 in [14] we see that $\varinjlim_k M(e, \mathbf{1}, K_\mu, k))$ is isomorphic to $e\mathcal{L}(\mathrm{Ind}_{B_1}^{G_1}(\mu))$, so we have $M(e, K_\mu) = e\mathcal{L}(\mathrm{Ind}_{B_1}^{G_1}(\mu))$.

We define the classical subspace $M(e, K_\mu)^{\mathrm{cl}}$ to be $e\mathcal{L}(\mathrm{Ind}_{B_1}^{G_1}(\mu)^{\mathrm{cl}})$, where $\mathrm{Ind}_{B_1}^{G_1}(\mu)^{\mathrm{cl}}$ is the intersection of $\mathrm{Ind}_{B_1}^{G_1}(\mu)$ with the image of $\mathcal{C}^{\mathrm{pol}}(G_1, K) \otimes_K \mathcal{C}^{\mathrm{sm}}(G_1, K)$ under the natural multiplication map. In particular, $\mathrm{Ind}_{B_1}^{G_1}(\lambda)^{\mathrm{cl}} = V \otimes_K \text{sm-}\mathrm{Ind}_{B_1}^{G_1}(\mathbf{1})$.

Theorem 39. *If* $\lambda \in X(\mathbf{T})$ *is dominant and arithmetical then we have a Hecke-equivariant exact sequence*

$$0 \to M(e, K_\lambda)^{cl} \to M(e, K_\lambda) \to \bigoplus_{w \in W^{(1)}} M(e, K_{w\cdot\lambda}) \to \cdots \to \bigoplus_{w \in W^{(i)}} M(e, K_{w\cdot\lambda}) \to \cdots \to M(e, K_{w_0\cdot\lambda}) \to 0.$$

Proof. We first show that all the terms in the exact sequence in Theorem 30 are arithmetical. In the proof of Theorem 16 we showed that $w \cdot \lambda|_{\mathbf{Z_G}} = \lambda|_{\mathbf{Z_G}}$ for all $w \in W$. As λ is arithmetical and $\mathbf{Z}_{\mathbf{H}}(\mathfrak{o}_F) \subseteq \mathbf{Z}_{\mathbf{G}}(L)$, we see that $w \cdot \lambda$ is arithmetical for all $w \in W$. Since $\mathbf{Z}_{\mathbf{H}}(\mathfrak{o}_F)$ acts on $\mathrm{Ind}_{B_1}^{G_1}(\mu)$ via the same character that it acts on A_μ, i.e. μ, it follows that

$\mathrm{Ind}_{B_1}^{G_1}(w \cdot \lambda)$ is arithmetical for all $w \in W$. Finally, $V \otimes_K \text{sm-Ind}_{B_1}^{G_1}(\mathbf{1})$ injects into an arithmetical representation and is therefore also arithmetical.

As explained in the proof of Corollary 3.3.5 in [14], the functor $e\mathcal{L}$ on the category of arithmetic representations is the same as taking the image of an idempotent in a finite-dimensional matrix algebra over the group ring $K[G_1]$. It is hence exact, and applying it to the exact sequence in Theorem 30 we get the required exact sequence. Hecke-equivariance follows from the M-equivariance of the original sequence. $\square$

Apart from $V \otimes_K \text{sm-Ind}_{B_1}^{G_1}(\mathbf{1}) \to \mathrm{Ind}_{B_1}^{G_1}(\lambda)$, the maps in the exact sequence in Theorem 30 are made up of the maps $(u_{\theta_{\alpha,w\cdot\lambda}})_L : \mathrm{Ind}_{B_1}^{G_1}(w \cdot \lambda) \to \mathrm{Ind}_{B_1}^{G_1}(s_\alpha w \cdot \lambda)$ for $w \in W$ and $\alpha \in \Phi^+$ such that $l(s_\alpha w) = l(w) + 1$. Given such a w and α, we define $\theta^{\mathrm{aut}}_{\alpha,w\cdot\lambda}$ to be $e\mathcal{L}((u_{\theta_{\alpha,w\cdot\lambda}})_L) : M(e, K_{w\cdot\lambda}) \to M(e, K_{s_\alpha w\cdot\lambda})$. It follows that all the maps in the exact sequence in Theorem 39 after the first are made up from these $\theta^{\mathrm{aut}}_{\alpha,w\cdot\lambda}$. In particular, $M(e, K_\lambda) \to \bigoplus_{w\in W^{(1)}} M(e, K_{w\cdot\lambda})$ is $\bigoplus_{\alpha\in\Delta} \theta^{\mathrm{aut}}_{\alpha,\lambda}$, from which we deduce that for any $\lambda \in X(\mathbf{T})$ which is dominant and arithmetical, $f \in M(e, K_\lambda)$ is in $M(e, K_\lambda)^{\mathrm{cl}}$ if and only if $f \in \ker \theta^{\mathrm{aut}}_{\alpha,\lambda}$ for all $\alpha \in \Delta$.

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