Slopes of Compact Hecke Operators

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Prologue

Abstract

In this thesis we present the definitions of the space of automorphic forms over a definite quaternion algebra. These spaces are infinite dimensional unlike spaces of classical modular forms which are all known to be finite.

For a general prime number, p, we define a Hecke Operator on this space of automorphic forms. These operators correspond, via the Jacquet-Langlands correspondence, to the usual classical Hecke Operators. The point is that spaces of classical modular forms embed into our space of automorphic forms and study of the Hecke operators on this infinite space gives us information about the Hecke operators on the classical spaces.

We shall investigate the slopes of a particular operator, U_3. The *slopes* of an operator are the gradients of the line segments that comprise the Newton polygon of the characteristic power series of that operator. We prove that, for all κ in a certain disc in weight space, the slopes of U_3 are in arithmetic progression, starting with

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \ldots.$$

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Contents

Introduction and Brief History

Interest in the explicit computation of slopes was first sparked by the work Coleman [Col97], Emerton [Eme98], Smithline [Smi94] and Coleman-Stevens-Teitelbaum [CST98]. They were studying overconvergent modular forms and did not prove much beyond facts about the smallest slope.

For instance, in [Eme98], Emerton proved that for the annulus $1 > |u| \geq |64|$ in weight space, the minimal slope of a Hecke operator, U_2, is some explicit function of κ. On the other hand, Smithline proved in [Smi94], that the average slope of overconvergent 3-adic modular forms was $3^{i+1} - 1$ for certain integers i and that the Newton Polygon of U_3 on the space of 3-adic overconvergent modular forms lies on or above the parabola

$$3\frac{n(n-1)}{2} + 2n.$$

He was able to compute the minimal slope in a range of cases: tame level 1 and all weights. Unfortunately, all of these results were only estimates.

In [Kil02], Kilford proves that the operator U_2 has slopes in arithmetic progression for all weights κ of the form $x \mapsto x^k$ with $k \in \mathbb{Z}$, odd. Of course, by continuity, these results extend to all $k \in 1 + 2\mathbb{Z}_2$.

The results in this thesis, are the first ever to prove specific things about all the slopes for *all* κ in a disc in weight space. In particular, the results are the first explicit computations of discs in the eigencurve of arbitrarily large weight. However, they tell us nothing about classical modular forms directly. To do this, it is necessary to use a p-adic version of the Jacquet-Langlands correspondence (announced by Chenevier) to translate our results to results about all the slopes of a classical eigencurve over a disc in weight space.

Chapter 1

Definitions and Preliminary Material

In this chapter, we introduce all the preliminary definitions and notations that we require. We also review some elementary lemmas and theorems both from number theory and ring theory, as well as the definition of a compact operator and of Hecke operators.

1.1 Infinite Matrices

Here we introduce the notion of an "infinite matrix", and various manipulations on these matrices.

1.1 Definition. Let R be a ring[1] and suppose that $a_{i,j}$ is a doubly infinite sequence of elements of R with $0 \leq i, j, \in \mathbb{Z}$. Then, we may form the infinite array:

$$\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which we will refer to as a *matrix* or *infinite matrix.* Departing from convention, we start numbering the rows and columns at 0; this is notationally convenient, as we shall see later.

Addition of two such infinite matrices, $(a_{i,j})_{0\leq i,j\in\mathbb{Z}}$ and $(b_{i,j})_{0\leq i,j\in\mathbb{Z}}$ is pointwise; viz.

$$(c_{i,j}) := (a_{i,j}) + (b_{i,j})$$

where $c_{i,j} = a_{i,j} + b_{i,j}$ for all $0 \leq i, j, \in \mathbb{Z}$. This is well defined and commutative as for the case of "finite" matrices.

[1] All rings are commutative with a 1.

In general, matrix multiplication of infinite matrices is not well defined as there are problems with convergence in an arbitrary ring, R. However, it will be necessary to pre-multiply and post-multiply certain matrices by *diagonal* matrices which do not cause convergence problems. To this end, we introduce the following notation.

1.2 Notation. Given a sequence, α_n $(0 \leq n \in \mathbb{Z})$ of elements in a ring R, we shall write $\mathrm{diag}((\alpha_n)_{0\leq n\in\mathbb{Z}})$ or $\mathrm{diag}(\alpha_0, \alpha_1, \alpha_2, \ldots)$ to denote the infinite matrix

$$\begin{pmatrix} \alpha_0 & 0 & 0 & \ldots \\ 0 & \alpha_1 & 0 & \ldots \\ 0 & 0 & \alpha_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.1.1}$$

If α is an element of $R - \{0\}$, we write $\mathrm{D}(\alpha)$ for the infinite matrix $\mathrm{diag}((\alpha^n)_{0\leq n\in\mathbb{Z}})$[2], i.e.

$$\mathrm{D}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & \alpha & 0 & \ldots \\ 0 & 0 & \alpha^2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.1.2}$$

Multiplication of a given infinite matrix $A = (a_{i,j})$ by $\mathrm{D}(\alpha)$ on the left (respectively right) simply has the effect of multiplying each entry in row $n \in \mathbb{Z}_{\geq 0}$ (respectively column n) by α^n. Thus, with multiplication by $\mathrm{D}(\alpha)$ we do not have any issues concerning infinite sums in an arbitrary ring.

Given an infinite matrix $A = (a_{i,j})$, we consider the following formal power series,

$$H_A(x,y) := \sum_{0\leq i,j\in\mathbb{Z}} a_{i,j}x^i y^j \in R[[x,y]].$$

$H_A(x,y)$ is called the *generating function* of the matrix A. If A is sufficiently "nice", we can write $H_A(x,y)$ as the (formal) quotient of two bivariate polynomials, viz.:

$$H_A(x,y) = \frac{F_A(x,y)}{G_A(x,y)}$$

with $F_A(x,y), G_A(x,y) \in R[x,y]$. Whenever such F_A and G_A exist, we call H_A the *rational function* of A.

Conversely, given two polynomials $F(x,y), G(x,y) \in R[x,y]$ such that the inverse of $G(x,y)$ exists in $R[[x,y]]$, we may consider the formal series expansion

$$\frac{F(x,y)}{G(x,y)} = \sum_{0\leq i,j\in\mathbb{Z}} b_{i,j}x^i y^j \in R[[x,y]],$$

[2] Here, by α^0 we, of course, mean 1.

(with $b_{i,j} \in R$) and form the infinite matrix

$$B = \begin{pmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \cdots \\ b_{1,0} & b_{1,1} & b_{1,2} & \cdots \\ b_{2,0} & b_{2,1} & b_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The function $\frac{F(x,y)}{G(x,y)}$ is then the generating function of B.

It is trivial to prove:

1.3 Proposition. *Let $A = (a_{i,j})$ be an infinite matrix over a ring R with generating function $H_A(x,y)$. Let $\alpha, \beta \in R - \{0\}$. Then, the generating function of $\mathrm{D}(\alpha) A \,\mathrm{D}(\beta)$ is $H_A(\alpha x, \beta y)$.* ■

Proposition 1.3 will play a crucial role in our calculations in the sequel and our main result will be proved using the rational functions of certain matrices.

1.2 Compact operators

We shall follow [Ser62] in the exposition of the theory.

Let p be a prime number and K be a complete subfield of $\mathbb{C}_p$ with norm denoted $|\cdot|$. Write A for the valuation ring of K, i.e. the set $\{x \in K : |x| \leq 1\}$, and $\mathfrak{m}$ for the maximal ideal of A. Let v_p denote the valuation on K normalised such that $v_p(p) = 1$.

1.4 Definition. A *Banach space over K* is a complete, normed vector space over K whose norm, $|\cdot|$, satisfies the ultrametric inequality

$$|x + y| \leq \max(|x|, |y|).$$

1.5 Definition. Given an index set I and a set of elements $x_i \in K, i \in I$, we say that x_i *tends to zero* and write $x_i \to 0$, if for all $\varepsilon > 0$ there exist only finitely many $i \in I$ such that $|x_i| > \varepsilon$.

Let I be an index set and let $c(I)$ denote the set of sequences $x = (x_i)_{i \in I}$, with $x_i \in K$ such that $x_i \to 0$. Define a norm

$$|x| = \sup_{i \in I} |x_i|.$$

It is easy to verify that this norm endows $c(I)$ with the structure of a Banach space.

1.6 Definition. Let E be a Banach space over K and suppose that there exists a set $\{e_i : i \in I\}$ of elements of E, I an index set, such that the following condition holds: Every $x \in E$ can be written uniquely as

$$\sum_{i \in I} x_i e_i$$

with $x_i \to 0$ and $|x| = \sup_{i\in I} |x_i|$. Then $(e_i)_{i\in I}$ is called an *orthonormal basis* for E.

Existence of orthonormal bases is straightforward to prove.

We shall restrict ourselves to spaces of the type $c(I)$.

Suppose that E and F are two Banach spaces over K. Write $\mathfrak{L}(E,F)$ for the vector space of continuous, linear maps from E to F. We equip $\mathfrak{L}(E,F)$ with the usual norm, viz: for $u \in \mathfrak{L}(E,F)$

$$|u| := \sup_{x\neq 0} \frac{|u(x)|}{|x|}.$$

This norm gives $\mathfrak{L}(E,F)$ the structure of a Banach space.

Write $E = c(I)$ (for an index set I) and assume that $(e_i)_{i\in I}$ is an orthonormal basis of E. If $u \in \mathfrak{L}(E,F)$, put $f_i = u(e_i)$. The f_i form a bounded family of elements of F. We have

1.7 Proposition ([Ser62], Proposition 3). *The map which associates an element $u \in \mathfrak{L}(E,F)$ to the sequence $(u(e_i))_{i\in I}$ is an isomorphism of the Banach space $\mathfrak{L}(E,F)$ with the space of bounded sequences $(f_i)_{i\in I}$ of F, equipped with the norm* $\sup_{i\in I} |f_i|$. ■

Explicitly, Proposition 1.7 gives us an example of an orthonormal basis for F, and so F can be identified with $c(J)$ for some index set J. We make this identification. The elements f_i can then be written as $(n_{ji})_{j\in J}$[3] with $n_{ji} \in K$, $|n_{ji}|$ bounded and $n_{ji} \to 0$ for fixed i and $j \to \infty$. We have,

$$|u| = \sup_{i,j} |n_{ji}|$$

If $x = (x_i)$ is an element of $c(I)$, we have $u(x) = (y_j)$ with

$$y_j = \sum_{i\in I} n_{ji} x_i.$$

We shall study the infinite matrix (n_{ji}) of u with respect to the orthonormal bases of E and F; note that the rows of this matrix are indexed by $j \in J$.

1.8 Definition. Let $\mathfrak{F}(E,F)$ denote the subspace of $\mathfrak{L}(E,F)$ consisting of the continuous, linear maps that are of finite rank. We say that u is *compact*[4] if u belongs to the closure of $\mathfrak{F}(E,F)$. Write $\mathfrak{C}(E,F)$ for the closure of $\mathfrak{F}(E,F)$.

Define $r_j(u) = \sup_{i\in I} |n_{ji}|$. We have

1.9 Proposition ([Ser62], corollary to Proposition 4). *The following are equivalent:*

1. *u is compact.*
2. *$r_j(u)$ tends to zero.* ■

[3] Note that the reversed subscripts compared with [Ser62] — this is a notational convenience.

[4] In [Ser62], the archaic terminology *completely continuous* is used.

A useful, special case of this proposition is,

1.10 Corollary. *Suppose that the matrix (n_{ji}) of u has all its entries in $\mathcal{O}_K$. If* $\mathrm{D}\left(\frac{1}{p}\right)(n_{ji})$ *has all its entries in $\mathcal{O}_K$, then u is compact.* ■

We will often say that a matrix is compact, if it is the matrix of a compact operator with respect to some basis.

We now introduce the notion of the *characteristic power series* of an operator.

Let L be a free module over a ring R, and suppose that f is an endomorphism of L such that $f(L)$ is contained in M — a free sub-module of L of finite type, and a direct factor of L (e.g. a submodule of L generated by a finite subset of the basis of L). Let $f_M : M \to M$ be the restriction of f to M. The polynomial $\det(1 - tf_M)$ is well defined, and it is straightforward to show that it is independent of the choice of M; write $\det(1 - tf)$ instead. Set

$$\det(1 - tf) = 1 + c_1 t + \ldots + c_m t^m + \ldots.$$

Suppose that $(e_i)_{i \in I}$ is a basis for L and that (n_{ji}) is the matrix of f with respect to this basis.[5] We can give the c_m explicitly as follows: if S is a finite subset of I, and σ is a permutation of S write

$$n_{S,\sigma} = \prod_{i \in S} n_{\sigma(i),i} \quad \text{and} \quad c_S = \sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) n_{S,\sigma}$$

where $\mathrm{Sym}(S)$ is the group of permutations of S and $\mathrm{sgn}(\sigma)$ is the signature of the permutation σ. Thus,

$$c_m = (-1)^m \sum_{\substack{S \subseteq I \\ |S| = m}} c_S. \tag{1.2.3}$$

Now we return to compact operators. Assume that u is a compact operator on E and that $|u| \leq 1$. If E_0 is the set of elements of E of norm at most 1, we have that $u(E_0) \subseteq E_0$. Assume that $\mathfrak{a}$ is a non-zero ideal of A contained in $\mathfrak{m}$. The operator u now defines, by passing to the quotient, an endomorphism $u_{\mathfrak{a}}$ of $E_{\mathfrak{a}} := E_0/\mathfrak{a}E_0$; the images of $(e_i)_{i \in I}$ in $E_{\mathfrak{a}}$ form a basis (in the algebraic sense) of $E_{\mathfrak{a}}$ considered as an $A/\mathfrak{a}$-module. If (n_{ji}) denotes the matrix of u with respect to $(e_i)_{i \in I}$ and if

$$r_j(u) = \sup_{i \in I} |n_{ji}|$$

we have that $r_j(u) \to 0$, and there exists a finite subset $T(\mathfrak{a})$ of I such that $n_{ji} \in \mathfrak{a}$ if $j \in I - T(\mathfrak{a})$. It follows that the image of $u_{\mathfrak{a}}$ is contained in a submodule of $E_{\mathfrak{a}}$ of finite type and that the polynomial $\det(1 - tu_{\mathfrak{a}})$ is well defined. The coefficients of $\det(1 - tu_{\mathfrak{a}})$ lie in $A/\mathfrak{a}$. As $\mathfrak{a}$ varies, the polynomials form a projective system and their limit is a formal power

[5] Note that $I = J$.

series denoted $\det(1-tu)$, whose coefficients tend to zero; this property does not depend on the product tu. If now, u is *any* compact endomorphism of E, we can choose a scalar c such that $|cu| \leq 1$ and thus $\det(1-tcu)$ is defined; and hence we may define $\det(1-tu)$ to be $f\left(\frac{t}{c}\right)$, where $f(t) = \det(1-t(cu))$.

1.11 Definition. $\det(1-tu)$, as defined above, is called the *characteristic power series* of u.

We shall also refer to $\det(1-tu)$ as the characteristic power series of the matrix (n_{ji}) of u.

1.12 Proposition ([Ser62], Proposition 7). *Let $u : E \to E$ be a compact operator.*

1. *If the matrix of u with respect to the orthonormal basis $(e_i)_{i\in I}$ is (n_{ji}) then*

$$\det(1-tu) = \sum_{m=0}^{\infty} c_m t^m$$

   with c_m defined as in (1.2.3).

2. *The series $\det(1-tu)$ is an entire function of t, i.e. the radius of convergence is infinite.*

3. *If $u_n \to u$ with $u_n \in \mathfrak{C}(E,E)$ for all $n \in \mathbb{N}$, then $\det(1-tu_n)$ tends to $\det(1-tu)$ coefficient-wise.*

4. *If u is of finite rank, then $\det(1-tu)$ coincides with the polynomial defined earlier.*

■

1.13 Remark. Parts (3) and (4) of Proposition 1.12 show that $\det(1-tu)$ does not depend on the *norm* of E, but only on the topology.

We also have,

1.14 Proposition ([Ser62], Corollary 2 to Proposition 7). *Suppose that $u \in \mathfrak{C}(E,E)$ and $v \in \mathfrak{L}(E,E)$. Then,*

$$\det(1-tu\circ v) = \det(1-tv\circ u).$$

(Note that this formula makes sense, since $u\circ v, v\circ u \in \mathfrak{C}(E,E)$.) ■

Lastly,

1.15 Lemma ([Ser62], Lemma 2). *Let $I = I' \cup I''$ be a partition of I. Assume that u is a compact endomorphism of $E = c(I)$ sending $E' = c(I')$ to itself. Let u' be the restriction* *of u to E', and let u'' be the endomorphism of $E'' = c(I'')$ defined by passing to the quotient by u. Then u' and u'' are compact and*

$$\det(1 - tu) = \det(1 - tu')\det(1 - tu'').$$

■

This result essentially says that if we split our space E into two subspaces, then the characteristic power series of u on the whole space is the product of the characteristic power series of the restriction of u to each subspace.

1.3 Newton Polygons

We follow [Kob84] in our exposition.

Assume that $f(z) = \sum_{i=0}^{\infty} a_i z^i$ is a power series with coefficients in K and assume that $a_0 = 1$. For $0 \leq n \in \mathbb{Z}$ define $f_n(z) = \sum_{i=0}^{n} a_i z^i$. Consider the following set of points in the real plane, $\mathbb{R}^2$,

$$P = \{(0,0), (1, v_p(a_1)), (2, v_p(a_2)), \ldots, (n, v_p(a_n))\}.$$

If for any i, a_i is zero, we omit that point, or regard it as being "at infinity" if we like to think of $\mathbb{R}^2$ being the Riemann sphere.

Recall that a set $S \subseteq \mathbb{R}^2$ is *convex* if for all pairs of points Q and R inside S, the entire line segment $\{\lambda Q + (1-\lambda)R : 0 \leq \lambda \leq 1\}$ is also contained in S. The *convex hull for P* is the smallest convex polygon, S, that contains every point of P; that is, there is no other convex polygon L with $P \subseteq L \subset S$. The *Newton Polygon* of f_n, written $\mathrm{NP}(f_n)$, is the *lower convex hull* of the set of points, P, i.e., the subset of line segments of S with the property that every point of P lies on or above that line segment.

Suppose that $\mathrm{NP}(f_n)$ consists of the line segments whose end points are $P_0 = (x_0, y_0) = (0,0), P_1 = (x_1, y_1), \ldots, P_m = (x_m, y_m)$. Note that $x_m \in \mathbb{Z}$ since all the x co-ordinates of the points of P are, and in general $m \leq n$. Clearly, the gradient, or slope, of the i-th line segment ($1 \leq i \leq m$) is,

$$g_i := \frac{y_i - y_{i-1}}{x_i - x_{i-1}}.$$

Given $1 \leq h \leq n$, we can find unique i_h such that $x_{i_h - 1} < h \leq x_{i_h}$, and the *$h$-th slope* of $\mathrm{NP}(f_n)$ is defined to be g_{i_h}. Of course, in general there may be two or more consecutive h's for which the h-th slope is the same.

All this makes sense for f_n. For f we define the Newton polygon of f to be the "limit" of the Newton polygons for the sequence f_n.

1.16 Lemma ([Kob84], corollary to Theorem 14). *Suppose that the Newton polygon of f has a line segment of finite length $N = x_i - x_{i-1}$ of slope g_i. Then, there are precisely N elements x such that $f(x) = 0$ and $v_p(x) = -g_i$.* ■

We shall apply all these notions to $\det(1 - tu)$, where u is a *Hecke operator* which we define below.

The main point about slopes is,

1.17 Lemma. *Let u be a compact operator on the Banach space E. Then the slopes of the Newton polygon of u are precisely the p-adic valuations of the eigenvalues of u.* ■

1.4 Preliminaries

As before, let p denote a prime number and v_p denote the valuation on $\mathbb{Q}$ associated to p normalised so that $v_p(p) = 1$. We shall also write v for the equivalence class of valuations on $\mathbb{Q}$ (i.e. the place) containing v and refer to p and v as the *prime*, *place* or *valuation.* As usual, write $\mathbb{Q}_p$ or $\mathbb{Q}_v$ for the completion of $\mathbb{Q}$ at p (or equivalently v), and write $\mathbb{Z}_p$ or $\mathbb{Z}_v$ for the ring of integers of $\mathbb{Q}_p$. Unless stated otherwise, all valuations on $\mathbb{Q}$ will be normalised in the usual way.

Let D be the discriminant 2 quaternion algebra over $\mathbb{Q}$ and write $D = \mathbb{Q}(\mathfrak{i}, \mathfrak{j})$. Take $\mathcal{O}_D = \mathbb{Z}\left[\mathfrak{i}, \mathfrak{j}, \frac{1}{2}(1 + \mathfrak{i} + \mathfrak{j} + \mathfrak{k})\right]$ as our fixed maximal order of D. In the calculations in the sequel, we shall be using explicit isomorphisms between quaternion and matrix algebras, and Hensel's lemma will provide some of the data we need for these isomorphisms. In particular, we have the following result whose proof is an easy exercise.

1.18 Lemma. *Suppose L is a field of characteristic zero. Then, $D \otimes_{\mathbb{Q}} L \cong M_2(L)$ if and only if there exist $\nu, \xi \in L$ such that $\nu^2 + \xi^2 = -1$.* ■

For v a place of $\mathbb{Q}$, let $D_v = D \otimes_{\mathbb{Q}} \mathbb{Q}_v$, and let S denote the set of places (including infinite ones) that do not split D, i.e.

$$S = \{v \text{ place of } \mathbb{Q} : D_v \ncong \mathrm{M}_2(\mathbb{Q}_v)\}.$$

Let S_f denote the subset of S consisting of the finite places,

$$S_f = \{v \text{ finite place of } \mathbb{Q} : D_v \ncong \mathrm{M}_2(\mathbb{Q}_v)\}.$$

It is straightforward to verify, that $\mathbb{Q}_q$ for all odd primes q satisfies the conditions of Lemma 1.18, and that

1.19 Lemma. *There are no elements $\nu, \xi \in \mathbb{Q}_2$ such that $\nu^2 + \xi^2 = -1$.* ■

Hence, $S = \{2, \infty\}$ and $S_f = \{2\}$.

For each prime q not in S, choose a fixed isomorphism

$$(\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_q)^\times \cong \mathrm{M}_2(\mathbb{Z}_q)^\times = \mathrm{GL}_2(\mathbb{Z}_q),$$

and extend to

$$\mathcal{O}_{D,q} := \mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_q \cong \mathrm{M}_2(\mathbb{Z}_q), \text{ and}$$
$$D \otimes_{\mathbb{Q}} \mathbb{Q}_q \cong \mathrm{M}_2(\mathbb{Q}_q).$$

in the natural way. We shall regard all of these isomorphisms as identifications, especially when dealing with certain groups $U_0(p^n)$ and $U_1(p^n)$, which shall be defined below.

One easily verifies that for all odd primes q the map $D_q \to \mathrm{M}_2(\mathbb{Q}_q)$ given by

$$a + b\mathfrak{i} + c\mathfrak{j} + d\mathfrak{k} \mapsto \begin{pmatrix} a + b\nu_q + d\xi_q & b\xi_q - c - d\nu_q \\ b\xi_q + c - d\nu_q & a - b\nu_q - d\xi_q \end{pmatrix}$$

is an isomorphism where $\nu_q, \xi_q \in \mathbb{Z}_q$ satisfy $\nu_q^2 + \xi_q^2 = -1$. We take this map as our fixed isomorphism for each q.[6] Whenever useful, we shall denote this map by θ, or by θ_q.

Write $\mathbb{A}$ for the adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ for the finite adeles of $\mathbb{Q}$. Thus,

$$\mathbb{A} = \left\{ (x_v) \in \prod_{v \nmid \infty} \mathbb{Q}_v \times \prod_{v \mid \infty} \mathbb{Q}_v : x_v \in \mathbb{Z}_v \text{ for all but finitely many } v \nmid \infty \right\},$$
$$\text{and } \mathbb{A}_f = \left\{ (x_v) \in \prod_{v \nmid \infty} \mathbb{Q}_v : x_v \in \mathbb{Z}_v \text{ for all but finitely many } v \right\}.$$

$\mathbb{A}$ (respectively $\mathbb{A}_f$) is a topological ring, and is given a topology such that $\prod_{v \nmid \infty} \mathbb{Z}_v \times \prod_{v \mid \infty} \mathbb{Q}_v$ (respectively $\prod_{v \nmid \infty} \mathbb{Z}_v$) is open with its usual topology. Note the natural embedding of $\mathbb{Q}$ diagonally into $\mathbb{A}$ and $\mathbb{A}_f$, viz. $\mathbb{Q} \ni x \mapsto (x, x, x, x, \ldots)$.

We form the tensor products

$$D_{\mathbb{A}} = D \otimes_{\mathbb{Q}} \mathbb{A}, \text{ and}$$
$$D_f = D \otimes_{\mathbb{Q}} \mathbb{A}_f.$$

$D_{\mathbb{A}}$ and D_f are also topological rings and they inherit their topologies from $\mathbb{A}$ and $\mathbb{A}_f$ respectively. We note at once that for $(x_q) \in D_f^\times$, $x_q \in \mathrm{GL}_2(\mathbb{Z}_q)$ for all but finitely many $q \notin S_f$.

Just as $\mathbb{Q}$ embeds diagonally into $\mathbb{A}$ and $\mathbb{A}_f$, D embeds diagonally into $D_{\mathbb{A}}$ and D_f. It will be convenient to identify an element of D with its diagonal image in D_f and we shall do this frequently without further mention.

[6] Of course, this isomorphism depends on the choice of the pair, (ν_q, ξ_q) which may not be unique. We choose any suitable fixed $\nu_q, \xi_q \in \mathbb{Z}_q$.

Recall that we have the norm map, $N : D \to \mathbb{Q}$ sending $d \in D$ to $d\bar{d}$. We may extend N to D_f and preserve the diagonal embedding of D; for $x = (x_v) \in D_f$ we have,

$$N(x) = (N(x_v))$$

and in fact, one can check that $N = \det \circ \theta_v$.

Our main interest is in $D_f^\times$, the unit group of D_f. This is a topological group, its topology being the subspace topology when embedded in $D_f \times D_f$ under the map $D_f^\times \hookrightarrow D_f \times D_f$ sending $x \mapsto (x, x^{-1})$; $D_f \times D_f$ is given the product topology here.

Let $U \leq D_f^\times$ be an open, compact subgroup of $D_f^\times$. It is straightforward to see that $D^\times \backslash D_f^\times / U$ is a finite set (as it is discrete and compact) and hence it follows that

$$D_f^\times = \coprod_{i \in I} D^\times c_i U$$

where I is a finite index set and c_i are elements of $D_f^\times$.

For each $i \in I$, put $\Gamma_i := c_i^{-1} D^\times c_i \cap U$. We shall return to these groups later.

1.20 Definition. Assume that $p \neq 2$, and $n \in \mathbb{N}$. We write,

$$U_0(p^n) = \prod_{\substack{q \text{ finite} \\ \text{primes of } \mathbb{Z}}} V_q, \text{ and}$$
$$U_1(p^n) = \prod_{\substack{q \text{ finite} \\ \text{primes of } \mathbb{Z}}} W_q$$

where

$$V_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p^n \right\} \text{ and}$$
$$W_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^n \right\},$$

where $*$ indicates the absence of a congruence condition, and for $q \neq p$ we have two cases:

1. $q \neq 2$: $W_q = V_q = \mathrm{GL}_2(\mathbb{Z}_q)$.
2. $q = 2$: we simply take W_q and V_q to be the group of units in any fixed maximal order of D_2.

We can also define $U_0(1)$ by taking $n = 0$ and in the definition of $U_0(1)$; thus

$$U_0(1) = U_0(p^0) = \prod_{\substack{q \text{ finite} \\ \text{primes of } \mathbb{Z}}} X_q,$$

where $X_q = \mathrm{GL}_2(\mathbb{Z}_q)$ for all $q \neq 2$ and X_2 is, again, the group of units in any fixed maximal order of D_2.

1.21 Proposition. *1. For all $n \in \mathbb{N}$, $U_0(p^n)$ and $U_1(p^n)$ are open compact subgroups of $D_f^\times$.*

2. For all $n \in \mathbb{N}$, $U_1(p^n)$ is a subgroup of $U_0(p^n)$. ∎

As $U_1(p^n)$ is a subgroup of $U_0(p^n)$, observe that $D^\times g U_1(p^n) \subset D^\times g U_0(p^n)$ for any $g \in D_f^\times$.

1.22 Lemma (cf. Theorem 2, Section 3, [Buzc]).

$$D_f^\times = D^\times U_0(1). \tag{1.4.4}$$

Proof. The shortest way is to use the Jacquet-Langlands correspondence: we know that there are no cusp forms of weight 2, and hence there are no 2-new cusp forms of weight 2. But the space of classical 2-new cusps forms is isomorphic, as a Hecke module, to the space of classical automorphic forms for D of level 1, weight 2. The dimension of this latter space is the (number of cosets) $-$ 1. Since the former space is zero, we obtain one coset. ∎

We are interested in finding the decomposition of $D_f^\times$ into $\coprod_{i \in I} D^\times c_i U$, or rather the representatives c_i, in the cases $U = U_0(p^n)$ and $U_1(p^n)$. This is achieved through a series of bijections, which we now detail for the case $U = U_1(p^n)$, $n \in \mathbb{N}$; the calculations for $U_0(p^n)$ are exactly similar and no more illuminating. We have,

$$\begin{aligned}
D^\times \backslash D_f^\times / U &= D^\times \backslash D^\times U_0(1)/U && (1.4.5)\\
&= D^\times \cap U_0(1) \backslash U_0(1)/U && (1.4.6)\\
&= \mathcal{O}_D^\times \backslash U_0(1)/U && (1.4.7)\\
&= \mathcal{O}_D^\times \backslash \operatorname{GL}_2(\mathbb{Z}_p)/H_1 && (1.4.8)\\
&= \mathcal{O}_D^\times \backslash \operatorname{GL}_2(\mathbb{Z}/p^n)/H_2 && (1.4.9)\\
&= \mathcal{O}_D^\times \backslash \operatorname{SL}_2(\mathbb{Z}/p^n)/H_3 && (1.4.10)
\end{aligned}$$

where

$$\begin{aligned}
H_1 &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \operatorname{GL}_2(\mathbb{Z}_p) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p^n \right\},\\
H_2 &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \operatorname{GL}_2(\mathbb{Z}/p^n) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p^n \right\}, \text{ and}\\
H_3 &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \operatorname{SL}_2(\mathbb{Z}/p^n) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p^n \right\}.
\end{aligned}$$

(1.4.5) holds as a consequence of (1.4.4); for (1.4.6) we need the following lemma from group theory:

1.23 Lemma (cf. [AB95], "First Isomorphism Theorem", page 11.). *Let G be a group. Suppose that H and K are subgroups of G such that $G = KH$. Then,*

$$K \backslash G = (H \cap K) \backslash H.$$

Proof. It is readily verified that the map $Kh \mapsto (H \cap K)\,h$ is a bijection. ■

We apply this lemma with $G = D^\times U_0(1)$, $K = D_f^\times$ and $H = U_0(1)$.

One easily verifies that $\mathcal{O}_D^\times = D^\times \cap U_0(1)$ and (1.4.7) follows, and (1.4.8) holds since at all primes $q \neq p$, everything is trivial.

(1.4.9) and (1.4.10) follow simply from set bijections. For (1.4.9),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} H_1 \rightarrow \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} H_2$$

where $\overline{u}$ is the image of u in $\mathbb{Z}/p^n$ and $u \in \{a, b, c, d\} \subset \mathbb{Z}$. For (1.4.10), the map is:

$$hH_2 \mapsto hH_3.$$

It is straightforward to verify that each of these maps is a bijection.

This form is still not conducive to calculating the c_i's. To this end, we briefly recall the theory of G-sets. Recall that if G is a group, a *G-set*, X, is a non-empty set such that there is a map $G \times X \rightarrow X$ which sends $(g, x) \in G \times X$ to $g.x \in X$, subject to the conditions:

1. $1_G.x = x$ for all $x \in X$

2. $g_1.(g_2.x) = (g_1 g_2).x$ for all $g_1, g_2 \in G$ and for all $x \in X$.

For $x \in X$, the *orbit of* x is the set $G.x := \{g.x : g \in G\}$, and the *stabilizer of* x is the set $G_x := \{g \in G : g.x = x\}$. One can easily verify that G_x is a (not necessarily normal) subgroup of G. Moreover there exists a bijection between $G.x$ and G/G_x, the set of cosets of G_x in G, sending $g.x \mapsto gG_x$.

Now take $G = \mathrm{SL}_2\left(\mathbb{Z}/p^n\right)$ and $X = \left\{ \binom{x_1}{x_2} : x_1, x_2 \in \mathbb{Z}/p^n \right\}$. The group G acts on X on the left via the usual multiplication of a vector by a matrix.

Consider $x = \binom{1}{0} \in X$.

1.24 Proposition. *We have*

$$G_x = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/p^n) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod p^n \right\}.$$

■

Thus,

$$\mathrm{SL}_2(\mathbb{Z}/p^n)/\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/p^n) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p^n \right\} = G/G_x$$

and this bijects with $G.x$. The following proposition tells us what $G.x$ is:

1.25 Proposition. *We have*

$$G.x = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X : \text{ at least one of } x_1, x_2 \text{ is in } (\mathbb{Z}/p^n)^\times \right\}.$$

■

Hence,

$$D^\times \backslash D_f^\times / U = \mathcal{O}_D^\times \backslash G.x. \tag{1.4.11}$$

Now $\mathcal{O}_D^\times$ injects into $\mathrm{SL}_2(\mathbb{Z}_p)$ and this surjects into $\mathrm{SL}_2(\mathbb{Z}/p^n)$; hence there is a map $\mathcal{O}_D^\times \to \mathrm{SL}_2(\mathbb{Z}/p^n)$. Therefore, $\mathcal{O}_D^\times$ acts in the usual way on $G.x$ via matrix multiplication on the left. Let m be the number of orbits and set $I = \{1, 2, \ldots, m\}$. Write

$$\mathcal{O}_D^\times \backslash G.x = \coprod_{i=1}^{m} \mathcal{O}_D^\times . s_i$$

where $s_1, \ldots, s_m$ are representatives of the orbits (recall that two orbits are either disjoint or equal). Each will be a column vector, say

$$s_i = \begin{pmatrix} \alpha_i \\ \gamma_i \end{pmatrix} \in G.x,$$

with $i \in I$. "Lift" $s_i = \left(\begin{smallmatrix} \alpha_i \\ \gamma_i \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbb{Z}_p)$ in the most obvious way. Put

$$c_{i,q} = \begin{cases} 1 \text{ if } q \neq p \\ \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \text{ if } q = p \end{cases} \tag{1.4.12}$$

It now follows that the c_i's are the representatives we require. Notice that $c_i \in U_0(1)$ for each $i \in I$. All of this will be exemplified in the sequel.

1.5 Automorphic forms

1.26 Definition (cf. [Ste]). Let $\mathcal{A}_p$ denote the Tate algebra

$$\mathbb{C}_p\langle z \rangle = \left\{ \sum_{k=0}^{\infty} a_k z^k : a_k \in \mathbb{C}_p \text{ and } a_k \to 0 \text{ } p\text{-adically as } k \to \infty \right\}.$$

In other words, $\mathcal{A}_p$ is the power series ring in one variable over $\mathbb{C}_p$ with the property that the coefficients a_k tend to zero p-adically as k tends to infinity.

$\mathcal{A}_p$ is a Banach space with norm given by

$$\|f\| = \sup_k |a_k|_3$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{A}_p$.

Next, we define a monoid and an action of this monoid on $\mathcal{A}_p$. We write $\mathcal{O}_p$ to denote the integers of $\mathbb{C}_p$ and $\mathfrak{m}_p$ for its maximal ideal. We assume from now on that p is an odd prime.

1.27 Definition (cf. [Ste]). Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_p^\times$ be a locally analytic character, i.e. a continuous group homomorphism. Given $\alpha \in \mathbb{N}$, let

$$\Sigma_\alpha = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_p) : p^\alpha \mid c, p \nmid d, \det(\gamma) \neq 0 \right\}.$$

The *weight κ action* of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_\alpha$ on $\mathcal{A}_p$ is given by the continuous $\mathbb{C}_p$-linear extension of the map sending

$$z^k \mapsto \frac{\kappa(cz+d)}{(cz+d)^2} \left(\frac{az+b}{cz+d} \right)^k \tag{1.5.13}$$

and given $f(z) \in \mathcal{A}_p$, we write $(f\|_\kappa \gamma)(z)$ for this action. Note, that by $\kappa(cz+d)$ we mean the power series expansion of $\kappa(cz+d)$ at zero.

It is an easy check that Σ_α is a monoid and that $\|_\kappa$ is a right-action of Σ_α on $\mathcal{A}_p$.

1.28 Definition. Let α belong to $\mathbb{N}$ and suppose that U is an open compact subgroup of $D_f^\times$. We say that U has *wild level* $\geq p^\alpha$ if the projection $U \to D_p^\times$, i.e. U_p, is contained in Σ_α.

1.29 Remark. The terminology of Definition 1.28 is *not* standard.

We are now ready to define the space of automorphic forms:

1.30 Definition (cf. Section 3 of [Buzc]). Fix $\alpha \in \mathbb{N}$ and κ as in Definition 1.27. Let U be an open, compact subgroup of $D_f^\times$ of wild level $\geq p^\alpha$. Let A be any right Σ_α-module. The *level U, weight κ space of automorphic forms* is the space,

$$\mathcal{L}(U, A) = \{\varphi : D_f^\times \to A : \varphi(dgu) = \varphi(g)\|_\kappa u_p \forall\, d \in D^\times, g \in D_f^\times, u \in U\}.$$

The main result of interest concerning $\mathcal{L}(U, A)$ is:

1.31 Lemma. *Let U be an open compact subgroup of $D_f^\times$ of wild level $\geq p^\alpha$ and suppose that $D_f^\times = \coprod_{i \in I} D^\times c_i U$. Then*

$$\mathcal{L}(U, A) \cong \bigoplus_{i \in I} A^{\Gamma_i}$$

as a $\mathbb{C}_p$-vector space, where $\Gamma_i := c_i^{-1} D^\times c_i \cap U$.

Proof. It is an easy check that the map $\mathcal{L}(U, A) \to \bigoplus_{i \in I} A^{\Gamma_i}$ given by

$$\varphi \mapsto (\varphi(c_i))_{i \in I}$$

is an isomorphism. Here A^{Γ_i} is the subset of A that is fixed by Γ_i. ∎

1.6 Hecke operators

We are now ready to define Hecke operators. We retain the notation of previous sections.

Let U be an open compact subgroup of $D_f^\times$. Fix $\alpha \in \mathbb{N}$ and κ as in Definition 1.27. Given $\varphi \in \mathcal{L}(U, A)$ we define a new right action of U on $\mathcal{L}(U, A)$: set

$$(\varphi|_\kappa u)(g) := \varphi\left(gu^{-1}\right) \|_\kappa u_p.$$

It is easily verified that this is a right action.

1.32 Definition. Let $v \in D_f^\times$ such that $v_p \in \Sigma_\alpha$. Then the double coset UvU may be written as a disjoint union

$$UvU = \coprod_{t \in T} Uv_t$$

where T is a finite set, and $v_t \in D_f^\times$. The *Hecke operator* is the map $[UvU] : \mathcal{L}(U, A) \to \mathcal{L}(U, A)$ given by

$$[UvU]\varphi := \sum_{t \in T} \varphi|_\kappa v_t.$$

Of particular interest are the *standard Hecke operators* defined as follows: Assume that l is an odd prime. Define ϖ_l to be the element of $\mathbb{A}_f$ which is l at l and 1 at all other places. Put $\eta_l = \left(\begin{smallmatrix} \varpi_l & 0 \\ 0 & 1 \end{smallmatrix}\right)$, i.e. η_l is trivial at all places except at l where it is $\left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

1.33 Definition. The *standard Hecke operators* are given by

$$T_l := [U\eta_l U] \quad \text{and} \quad S_l := [U\varpi_l U].$$

If $l = p$, it is traditional to write U_p for T_p.

We shall focus on the operators U_p acting on the space $\mathcal{L}(U, \mathcal{A}_p)$, where U is $U_0(p^n)$ or $U_1(p^n)$. Write $U_p = [U\eta_p U] = \left[\coprod_{t \in T} Uv_t\right]$.

Our main interest is in the matrix of U_p and our first goal is to explain how to obtain its matrix. Consider the following diagram

$$\begin{array}{ccc} \mathcal{L}(U, \mathcal{A}_p) & \xrightarrow{\varphi \mapsto (\varphi(c_i))_{i \in I}} & \bigoplus_{i \in I} \mathcal{A}_p^{\Gamma_i} \\ {\scriptstyle \varphi \mapsto (U_p \varphi)} \downarrow & & \downarrow {\scriptstyle \text{Map of interest}} \\ \mathcal{L}(U, \mathcal{A}_p) & \xrightarrow[U_p\varphi \mapsto ((U_p\varphi)(c_i))_{i \in I}]{} & \bigoplus_{i \in I} \mathcal{A}_p^{\Gamma_i} \end{array}$$

We are interested in the vertical map on the right-hand edge of the commutative diagram, i.e. the map $\bigoplus_{i \in I} \mathcal{A}_p^{\Gamma_i}$ to $\bigoplus_{i \in I} \mathcal{A}_p^{\Gamma_i}$ that sends

$$(\varphi(c_i))_{i \in I} \to ((U_p\varphi)(c_i))_{i \in I}.$$

We shall also call this map U_p. Let $\{e_k : k \in \mathcal{K}\}$, where $\mathcal{K}$ is some index set, be a topological basis for $\bigoplus_{i\in I} \mathcal{A}_p^{\Gamma_i}$. By Lemma 1.31, for all $k \in \mathcal{K}$, there exists a unique preimage, $\varphi_k \in \mathcal{L}(U, \mathcal{A}_p)$, of e_k. I.e.,

$$e_k = (\varphi_k(c_i))_{i\in I}.$$

Now U_p sends φ_k to $\sum_{t\in T} \varphi_k|_\kappa v_t$. So,

$$\begin{aligned} U_p(e_k) &= ((U_p\varphi_k)(c_i))_{i\in I} \\ &= \left(\sum_{t\in T} (\varphi_k|_\kappa v_t)\,(c_i)\right)_{i\in I} \end{aligned}$$

and the right hand side is some element of $\bigoplus_{i\in I} \mathcal{A}_p^{\Gamma_i}$, and hence it may be written uniquely as a linear combination of the e_k. Hence we obtain the matrix of U_p with respect to the topological basis $\{e_k : k \in \mathcal{K}\}$.

In this notation, U_p can be represented as $|I|^2$ endomorphisms, say $\varepsilon_{i,j}$, from $\mathcal{A}_p^{\Gamma_j}$ in $\bigoplus_{k\in I} \mathcal{A}_p^{\Gamma_k}$ to $\mathcal{A}_p^{\Gamma_i}$. This justifies writing,

$$(U_p\varphi)(c_i) = \sum_{j\in I} \varepsilon_{i,j}\varphi(c_j).$$

And using the definition of U_p, we may write

$$\begin{aligned} (U_p\varphi)(c_i) &= \sum_{j\in I} \varepsilon_{i,j}\varphi(c_j) \\ &= \sum_{t\in T} (\varphi|_\kappa v_t)(c_i) \text{ (definition of } U_p \text{)} \\ &= \sum_{t\in T} \varphi(c_i v_t^{-1})\|_\kappa v_{t,p} \text{ (definition of } |_\kappa \text{).} \end{aligned}$$

Here $v_{t,p}$ means the p-part of v_t.

Next, we decompose $c_i v_t^{-1}$ as $d(i,t)c(i,t)u(i,t)$ with $d(i,t) \in D^\times$, $c(i,t) \in \{c_0, c_1, c_2\}$ and $u(i,t) \in U$. See Lemmas 2.4 and 2.5 for details of an example. Therefore,

$$\begin{aligned} (U_p\varphi)(c_i) &= \sum_{t\in T} \varphi(c_i v_t^{-1})\|_\kappa v_{t,p} \\ &= \sum_{t\in T} \varphi(c(i,t))\|_\kappa (u(i,t)v_t)_p. \end{aligned}$$

We may think of $\varepsilon_{i,j}$ as being $\|_\kappa (u(i,t)v_t)_p$ if $c_j = c(i,t)$. On some occasions we will find that there is no j such that $c_j = c(i,t)$; here $\varepsilon_{i,j}$ is the zero endomorphism.

Chapter 2

U_3

2.1 U_3 Evaluated

We work in the case $p = 3$ and investigate the Hecke operator U_3, level $U_1(9)$. We focus on maps κ satisfying the following properties,

1. we can write $\kappa(4) = 4^t$, where $t = \frac{\log_3(\kappa(4))}{\log_3(4)}$.
2. $v_3(t) > 0$, so that $t \in \mathfrak{m}_3$,
3. and without loss of generality, $\kappa(-1) = 1$.

Here, $\mathcal{A}_3$ is simply,

$$\left\{ \sum_{k=0}^{\infty} a_k z^k : a_k \to 0 \text{ 3-adically as } k \to \infty \right\}.$$

We shall be working over the ring $\mathbb{Z}_3$, and making use of the isomorphism θ_3 of Chapter 1. To this end, we choose our elements ν_3, ξ_3 of $\mathbb{Z}_3$ such that $\nu_3^2 + \xi_3^2 = -1$ as follows: take ν_3 to be the square root of -2 in $\mathbb{Z}_3$ that is 508 mod 3^7 and $\xi_3 = 1$. It is readily verified that $(\nu_3)^2 = -2$. It is an easy exercise to verify that ν_3 has 3-adic expansion beginning with,

$$\nu_3 = 1 + 3 + 2 \cdot 3^2 + 2 \cdot 3^5 + 3^7 + 2 \cdot 3^{11} + 3^{12} + \dots$$

Recall that in $D = \mathbb{Q}(\mathfrak{i}, \mathfrak{j})$, our maximal order $\mathcal{O}_D$ is $\mathbb{Z}\left[\mathfrak{i}, \mathfrak{j}, \frac{1}{2}(1 + \mathfrak{i} + \mathfrak{j} + \mathfrak{k})\right]$, and so

$$\mathcal{O}_D^\times = \{\pm 1, \pm \mathfrak{i}, \pm \mathfrak{j}, \pm \mathfrak{k}, \pm u_1, \pm u_2, \pm u_3, \pm u_4, \pm u_5, \pm u_6, \pm u_7, \pm u_8\}$$

where

$$\begin{aligned} u_1 &= \tfrac{1}{2}\left(1+\mathfrak{i}+\mathfrak{j}+\mathfrak{k}\right), \\ u_2 &= \tfrac{1}{2}\left(-1+\mathfrak{i}+\mathfrak{j}+\mathfrak{k}\right), \\ u_3 &= \tfrac{1}{2}\left(1-\mathfrak{i}+\mathfrak{j}+\mathfrak{k}\right), \\ u_4 &= \tfrac{1}{2}\left(1+\mathfrak{i}-\mathfrak{j}+\mathfrak{k}\right), \\ u_5 &= \tfrac{1}{2}\left(1+\mathfrak{i}+\mathfrak{j}-\mathfrak{k}\right), \\ u_6 &= \tfrac{1}{2}\left(-1-\mathfrak{i}+\mathfrak{j}+\mathfrak{k}\right), \\ u_7 &= \tfrac{1}{2}\left(-1+\mathfrak{i}-\mathfrak{j}+\mathfrak{k}\right) \text{ and} \\ u_8 &= \tfrac{1}{2}\left(-1+\mathfrak{i}+\mathfrak{j}-\mathfrak{k}\right). \end{aligned}$$

The details may be found in [Vig80].

Our first goal is to find coset representatives in the decomposition. We have

$$D_f^\times = \coprod_{i\in I} D^\times c_i U_1(9).$$

We have

2.1 Theorem.

$$D_f^\times = D^\times c_0 U_1(9) \amalg D^\times c_1 U_1(9) \amalg D^\times c_2 U_1(9)$$

where

$$\begin{aligned} c_{0,p} &= 1 \ \forall\ p \\ c_{1,p} &= \begin{cases} 1 \ \textit{if } p \neq 3 \\ \left(\begin{smallmatrix} 5 & 0 \\ 0 & 2 \end{smallmatrix}\right) \ \textit{if } p = 3 \end{cases} \\ c_{2,p} &= \begin{cases} 1 \ \textit{if } p \neq 3 \\ \left(\begin{smallmatrix} 7 & 0 \\ 0 & 4 \end{smallmatrix}\right) \ \textit{if } p = 3 \end{cases} \end{aligned}$$

Proof. Most of the hard work has been done in Chapter 1, and all we need do is to compute the right hand side of (1.4.11); here $G = \mathrm{SL}_2(\mathbb{Z}/(9))$. The image of $\mathcal{O}_D^\times$ is

$$\pm 1 \mapsto \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm\,\mathfrak{i} \mapsto \pm\begin{pmatrix} 4 & 1 \\ 1 & 5 \end{pmatrix}, \quad \pm\mathfrak{j} \mapsto \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm\mathfrak{k} \mapsto \pm\begin{pmatrix} 1 & 5 \\ 5 & 8 \end{pmatrix}$$
$$\pm u_1 \mapsto \pm\begin{pmatrix} 3 & 7 \\ 8 & 7 \end{pmatrix}, \quad \pm\, u_2 \mapsto \pm\begin{pmatrix} 2 & 7 \\ 8 & 6 \end{pmatrix}, \quad \pm u_3 \mapsto \pm\begin{pmatrix} 8 & 6 \\ 7 & 2 \end{pmatrix}, \quad \pm u_4 \mapsto \pm\begin{pmatrix} 3 & 8 \\ 7 & 7 \end{pmatrix}$$
$$\pm u_5 \mapsto \pm\begin{pmatrix} 2 & 2 \\ 3 & 8 \end{pmatrix}, \quad \pm\, u_6 \mapsto \pm\begin{pmatrix} 7 & 6 \\ 7 & 1 \end{pmatrix}, \quad \pm u_7 \mapsto \pm\begin{pmatrix} 2 & 8 \\ 7 & 6 \end{pmatrix}, \quad \pm u_8 \mapsto \pm\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}.$$

In this case, $|G.x| = 72$, and an unilluminating calculation shows that $\mathcal{O}_D^\times \backslash G.x$ consists of three elements, i.e. there are three orbits. We take $I = \{0, 1, 2\}$. Suitable representatives of the orbits are,

$$s_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ s_1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix},\ s_2 = \begin{pmatrix} 7 \\ 0 \end{pmatrix},$$

and we "lift" each s_i:

$$s_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ s_1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix},\ s_2 = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix}.$$

We define c_i as in (1.4.12), and the theorem follows. ∎

Our next goal is to calculate the groups Γ_i. Recall, that $\Gamma_i = c_i^{-1} D^\times c_i \cap U$.

2.2 Lemma. $\Gamma_i = \{1\}$ *for all* $i = 0, 1, 2$.

Proof. The only place of $\mathbb{Q}$ where we have any control is at $q = 3$, where all elements of U are of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ mod 9.

$\Gamma_0 = D^\times \cap U$, and as U is contained in $U_0(1)$, Γ_0 is a subgroup of $\mathcal{O}_D^\times$. If $u \in U$, then $u \in \mathcal{O}_D^\times$ and $u_3 \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$, and from the proof of Theorem 2.1, we see that the only possibility for Γ_0 is the trivial group.

$\Gamma_1 = c_1^{-1} D^\times c_1 \cap U$. For $u \in \Gamma_1$ write $u = c_1^{-1} d c_1$ with $d \in D^\times$. It is then clear that $d \in \mathcal{O}_D^\times$. Hence, $u_3 = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 2 \end{smallmatrix}\right)^{-1} d_3 \left(\begin{smallmatrix} 5 & 0 \\ 0 & 2 \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 2 & 0 \\ 0 & 5 \end{smallmatrix}\right) d_3 \left(\begin{smallmatrix} 5 & 0 \\ 0 & 2 \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ mod 9. This implies that $d_3 \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ mod 9, and from the proof of Theorem 2.1, we find that the only possibility for d is $d = 1$. Thus, Γ_1 is the trivial group.

An exactly similar argument shows that Γ_2 is also trivial. ∎

Recall from Lemma 1.31, that $\mathcal{L}(U, \mathcal{A}_3) \cong \bigoplus_{i=0}^2 \mathcal{A}_3^{\Gamma_i}$. As Γ_i is trivial for all i, it follows that

$$\mathcal{L}(U, \mathcal{A}_3) \cong \bigoplus_{i=0}^{2} \mathcal{A}_3. \tag{2.1.1}$$

We take

$$\mathfrak{B}_1 = \left\{ \begin{pmatrix} e_i(z) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_j(z) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e_k(z) \end{pmatrix} : 0 \leq i, j, k \in \mathbb{Z} \right\}$$

as our topological basis for $\bigoplus_{i=0}^2 \mathcal{A}_3$, where $e_0(z) = 1$ and $e_h(z) = z^h$ for $h \in \mathbb{N}$.

The Hecke operator U_3 is defined (see Definition 1.32) as,

$$U_3 = [U \eta_3 U]$$

where

$$\eta_{3,q} = \begin{cases} 1 \text{ if } q \neq 3 \\ \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } q = 3 \end{cases}$$

We need now, to calculate the decomposition of $U\eta_3 U$ into a disjoint union of single cosets. We note that η_3 is trivial at all primes $q \neq 3$, so it suffices to prove the following elementary result.

2.3 Lemma. *Write* $G = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_3) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod 9 \right\}$. *Then,*

$$G \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} G = G \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \amalg G \begin{pmatrix} 3 & 0 \\ 9 & 1 \end{pmatrix} \amalg G \begin{pmatrix} 3 & 0 \\ 18 & 1 \end{pmatrix} \tag{2.1.2}$$

■

Thus, we have,

$$U\eta_3 U = \coprod_{t=0}^{2} U v_t$$

where $v_{t,p} = 1$ if $p \neq 3$ and

$$v_{t,3} = \begin{pmatrix} 3 & 0 \\ 9t & 1 \end{pmatrix} = 2 - (\nu_3 + \tfrac{9}{2} t \xi_3)\mathfrak{i} + \tfrac{9}{2} t \mathfrak{j} + (\tfrac{9}{2} t \nu_3 - \xi_3)\mathfrak{k},$$

for $t = 0, 1, 2$.

Recall that U_3 may be represented by $|I|^2 = 9$ endomorphisms. Our goal now is to compute these. From Chapter 1 we know that we may find these endomorphisms by evaluating $U_3\varphi$ at c_0, c_1 and c_2. Therefore,

$$\begin{aligned} (U_3\varphi)(c_i) &= \sum_{j=0}^{2} \varepsilon_{i,j} \varphi(c_j) \\ &= \sum_{t=0}^{2} (\varphi|_\kappa v_t)(c_i) \\ &= \sum_{t=0}^{2} \varphi(c_i v_t^{-1}) \|_\kappa v_{t,3} \\ &= \sum_{t=0}^{2} \varphi(c_i v_t^{-1}) \|_\kappa \left(\begin{smallmatrix} 3 & 0 \\ 9t & 1 \end{smallmatrix}\right). \end{aligned}$$

It will be convenient to identify an element of $D_f^\times$ with its 3-part whenever that element is trivial at all other place. Thus, $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ will simply refer to the element that is 1 at all primes not equal to 3, and $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ at 3; $d \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right) \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ will indicate the element that is the product of the quaternion $d \in D^\times$, the element that is 1 at all primes not equal to 3 and $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ at 3 and the element that is 1 at all primes not equal to 3 and $\left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ at 3.

The next step is to determine the cosets in the decomposition from Theorem 2.1 to which each of the 9 elements $c_i v_t^{-1}$ belong. We present the algorithm for solving this problem in the form of two lemmas:

2.4 Lemma. *There exists* $d \in D^\times$ *such that* $d^{-1} c_i v_t^{-1} \in U_0(1)$.

Proof. From (1.4.4) we can write $c_i v_t^{-1} = du$. Hence, $d^{-1} = uv_t c_i^{-1}$. Taking determinants (or what is the same thing, norms) gives us:

- *Determinants at $l \neq 2, 3$:* $\det\left(d_l^{-1}\right) = \det(u_l)\det(v_{t,l})\det\left(c_{i,l}^{-1}\right) \in \mathbb{Z}_l^\times$
- *"Determinants" i.e. norms at $l = 2$:* $\det\left(d_2^{-1}\right) = N(u_2)N(v_{t,2})N\left(c_{i,2}^{-1}\right) \in \mathbb{Z}_2^\times$
- *Determinants at $l = 3$:* $\det\left(d_3^{-1}\right) = \det(u_3)\det(v_{t,3})\det\left(c_{i,3}^{-1}\right) \in 3\mathbb{Z}_3^\times$

Hence, $\det\left(d_l^{-1}\right) = 3\mu$ with $\mu \in \mathbb{Z}_l^\times$ for all primes l. Thus $\mu = 1$ since $\mu \in \bigcap_l \mathbb{Z}_l^\times = \{\pm 1\}$ and since N (and det) maps D to $\mathbb{Q}_{\geq 0}$.

Certainly, $d^{-1} \in \mathcal{O}_{D,l}$ for all primes l. So it follows that $d^{-1} \in \mathcal{O}_D$. So writing $d^{-1} = a_0 + a_1\mathfrak{i} + a_2\mathfrak{j} + a_3\mathfrak{k}$ gives that $|a_r| \leq \sqrt{3} < 1.8$. Hence for $r = 0, 1, 2, 3$, $a_r \in \left\{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}\right\}$.

Next, we can write

$$\begin{aligned} d_3^{-1} &= u_3 v_{t,3} c_{i,3}^{-1} \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 9t & 1 \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & \theta \end{pmatrix}, \text{ say} \\ &= \begin{pmatrix} 3(\alpha + 3\beta t)\zeta & \beta\theta \\ 3(\gamma + 3\delta t)\zeta & \delta\theta \end{pmatrix} \\ &\equiv \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \bmod 3 \end{aligned}$$

since ζ and θ are units and at least one of β and δ are units.

All that remains is to check the possibilities (see Section B.1) and we find that $d^{-1} = -\frac{3}{2} - \frac{1}{2}\mathfrak{i} - \frac{1}{2}\mathfrak{j} + \frac{1}{2}\mathfrak{k}$, so that $d = -\frac{1}{2} + \frac{1}{6}\mathfrak{i} + \frac{1}{6}\mathfrak{j} - \frac{1}{6}\mathfrak{k}$. ■

2.5 Lemma. *Given $\tilde{u} \in U_0(1)$, there exist $\alpha \in \mathcal{O}_D^\times$, $i \in I$ and $u \in U$ such that $\tilde{u} = \alpha c_i u$.*

Proof. Given such $\tilde{u}$, we can write $\tilde{u} = \alpha c_i u$ for some $\alpha \in D^\times$, $i \in I$ and $u \in U$ by Theorem 2.1. But $\alpha = \tilde{u}u^{-1}c_i^{-1} \in U_0(1)$. So, $\alpha \in U_0(1) \cap D^\times = \mathcal{O}_D^\times$. ■

In view of Lemma 2.5, we simply test the possibilities $c_i^{-1}\alpha^{-1}\tilde{u}$ to see which lies in U.

From the calculations of Section B.1, which implement Lemmas 2.4 and 2.5, we find that

$$c_0 v_0^{-1} = \left(-\tfrac{1}{3} - \tfrac{1}{3}\mathfrak{i} + \tfrac{1}{3}\mathfrak{j}\right) \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{21}\nu_3 - \frac{1}{21} & \frac{2}{7} \\ 0 & -\frac{1}{4}\nu_3 - \frac{1}{4} \end{pmatrix}$$

$$c_1 v_0^{-1} = \left(\tfrac{1}{3} + \tfrac{1}{3}\mathfrak{i} - \tfrac{1}{3}\mathfrak{j}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{5}{3}\nu_3 + \frac{5}{3} & -4 \\ 0 & 2\nu_3 + 2 \end{pmatrix}$$

$$c_2 v_0^{-1} = \left(\tfrac{1}{3} + \tfrac{1}{3}\mathfrak{i} - \tfrac{1}{3}\mathfrak{j}\right) \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -\frac{7}{15}\nu_3 + \frac{7}{15} & -\frac{8}{5} \\ 0 & 2\nu_3 + 2 \end{pmatrix}$$

$$
\begin{aligned}
c_0 v_1^{-1} &= \left(-\tfrac{1}{6} - \tfrac{1}{2}\mathfrak{i} - \tfrac{1}{6}\mathfrak{j} - \tfrac{1}{6}\mathfrak{k}\right) \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{5}\nu_3 - \frac{3}{5} & -\frac{1}{10}\nu_3 + \frac{1}{5} \\ \frac{13}{6}\nu_3 + \frac{11}{6} & -\frac{3}{4}\nu_3 - \frac{1}{2} \end{pmatrix} \\
c_1 v_1^{-1} &= \left(-\tfrac{1}{6} - \tfrac{1}{2}\mathfrak{i} - \tfrac{1}{6}\mathfrak{j} - \tfrac{1}{6}\mathfrak{k}\right) \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{11}{14}\nu_3 - \frac{6}{7} & -\frac{1}{7}\nu_3 + \frac{2}{7} \\ \frac{49}{24}\nu_3 + \frac{7}{3} & -\frac{3}{4}\nu_3 - \frac{1}{2} \end{pmatrix} \\
c_2 v_1^{-1} &= \left(\tfrac{1}{6} + \tfrac{1}{2}\mathfrak{i} + \tfrac{1}{6}\mathfrak{j} + \tfrac{1}{6}\mathfrak{k}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{19}{2}\nu_3 + 12 & 2\nu_3 - 4 \\ -\frac{101}{6}\nu_3 - \frac{50}{3} & 6\nu_3 + 4 \end{pmatrix} \\
c_0 v_2^{-1} &= \left(-\tfrac{1}{6} + \tfrac{1}{6}\mathfrak{i} + \tfrac{1}{2}\mathfrak{j} + \tfrac{1}{6}\mathfrak{k}\right) \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -\frac{19}{30}\nu_3 - \frac{19}{15} & \frac{1}{10}\nu_3 + \frac{1}{5} \\ -\frac{17}{12}\nu_3 - \frac{1}{3} & \frac{1}{4}\nu_3 \end{pmatrix} \\
c_1 v_3^{-1} &= \left(-\tfrac{1}{6} + \tfrac{1}{6}\mathfrak{i} + \tfrac{1}{2}\mathfrak{j} + \tfrac{1}{6}\mathfrak{k}\right) \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -\frac{41}{42}\nu_3 - \frac{41}{21} & \frac{1}{7}\nu_3 + \frac{2}{7} \\ -\frac{31}{24}\nu_3 - \frac{5}{6} & \frac{1}{4}\nu_3 \end{pmatrix} \\
c_2 v_2^{-1} &= \left(\tfrac{1}{6} - \tfrac{1}{6}\mathfrak{i} - \tfrac{1}{2}\mathfrak{j} - \tfrac{1}{6}\mathfrak{k}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{79}{6}\nu_3 + \frac{79}{3} & -2\nu_3 - 4 \\ \frac{65}{6}\nu_3 + \frac{14}{3} & -2\nu_3 \end{pmatrix}.
\end{aligned}
$$

For brevity, we shall exemplify the calculations only in the case $(U_3\varphi)(c_0)$; the other cases are exactly similar.

Using the definition of U_3 and φ we have,

$$
\begin{aligned}
(U_3\varphi)(c_0) &= \varepsilon_{0,0}\varphi(c_0) + \varepsilon_{0,1}\varphi(c_1) + \varepsilon_{0,2}\varphi(c_2) = \sum_{t=0}^{2} \varphi(c_0 v_t^{-1}) \|_\kappa \left(\begin{smallmatrix} 3 & 0 \\ 9t & 1 \end{smallmatrix}\right) \\
&= \varphi\left(\left(-\tfrac{1}{3} - \tfrac{1}{3}\mathfrak{i} + \tfrac{1}{3}\mathfrak{j}\right) c_2 \begin{pmatrix} \frac{1}{21}\nu_3 - \frac{1}{21} & \frac{2}{7} \\ 0 & -\frac{1}{4}\nu_3 - \frac{1}{4} \end{pmatrix}\right) \Bigg\|_\kappa \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + \\
&\quad \varphi\left(\left(-\tfrac{1}{6} - \tfrac{1}{2}\mathfrak{i} - \tfrac{1}{6}\mathfrak{j} - \tfrac{1}{6}\mathfrak{k}\right) c_1 \begin{pmatrix} \frac{2}{5}\nu_3 - \frac{3}{5} & -\frac{1}{10}\nu_3 + \frac{1}{5} \\ \frac{13}{6}\nu_3 + \frac{11}{6} & -\frac{3}{4}\nu_3 - \frac{1}{2} \end{pmatrix}\right) \Bigg\|_\kappa \begin{pmatrix} 3 & 0 \\ 9 & 1 \end{pmatrix} + \\
&\quad \varphi\left(\left(-\tfrac{1}{6} + \tfrac{1}{6}\mathfrak{i} + \tfrac{1}{2}\mathfrak{j} + \tfrac{1}{6}\mathfrak{k}\right) c_1 \begin{pmatrix} -\frac{19}{30}\nu_3 - \frac{19}{15} & \frac{1}{10}\nu_3 + \frac{1}{5} \\ -\frac{17}{12}\nu_3 - \frac{1}{3} & \frac{1}{4}\nu_3 \end{pmatrix}\right) \Bigg\|_\kappa \begin{pmatrix} 3 & 0 \\ 18 & 1 \end{pmatrix} \\
&= \varphi(c_2)\Big\|_\kappa \left(\begin{pmatrix} \frac{1}{21}\nu_3 - \frac{1}{21} & \frac{2}{7} \\ 0 & -\frac{1}{4}\nu_3 - \frac{1}{4} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\right) + \\
&\quad \varphi(c_1)\Big\|_\kappa \left(\begin{pmatrix} \frac{2}{5}\nu_3 - \frac{3}{5} & -\frac{1}{10}\nu_3 + \frac{1}{5} \\ \frac{13}{6}\nu_3 + \frac{11}{6} & -\frac{3}{4}\nu_3 - \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 9 & 1 \end{pmatrix}\right) + \\
&\quad \varphi(c_1)\Big\|_\kappa \left(\begin{pmatrix} -\frac{19}{30}\nu_3 - \frac{19}{15} & \frac{1}{10}\nu_3 + \frac{1}{5} \\ -\frac{17}{12}\nu_3 - \frac{1}{3} & \frac{1}{4}\nu_3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 18 & 1 \end{pmatrix}\right)
\end{aligned}
$$

and comparing the first and last lines gives, upon simplification,

$$\begin{aligned}
\varepsilon_{0,0}\varphi(c_0) &= 0\\
\varepsilon_{0,1}\varphi(c_1) &= \varphi(c_1)\big\|_\kappa \begin{pmatrix} \frac{3}{10}\nu_3 & -\frac{1}{10}\nu_3+\frac{1}{5}\\ -\frac{1}{4}\nu_3+1 & -\frac{3}{4}\nu_3-\frac{1}{2}\end{pmatrix} +\\
&\qquad \varphi(c_1)\big\|_\kappa \begin{pmatrix} -\frac{1}{10}\nu_3-\frac{1}{5} & \frac{1}{10}\nu_3+\frac{1}{5}\\ \frac{1}{4}\nu_3-1 & \frac{1}{4}\nu_3\end{pmatrix}\\
\varepsilon_{0,2}\varphi(c_2) &= \varphi(c_2)\big\|_\kappa \begin{pmatrix} \frac{1}{7}\nu_3-\frac{1}{7} & \frac{2}{7}\\ 0 & -\frac{1}{4}\nu_3-\frac{1}{4}\end{pmatrix}.
\end{aligned}$$

Similarly, we obtain,

$$\begin{aligned}
\varepsilon_{1,0}\varphi(c_0) &= \varphi(c_1)\|_\kappa \left(\begin{smallmatrix} -5\nu_3+5 & -4\\ 0 & 2\nu_3+2\end{smallmatrix}\right)\\
\varepsilon_{1,1}\varphi(c_1) &= 0\\
\varepsilon_{1,2}\varphi(c_2) &= \varphi(c_2)\big\|_\kappa \begin{pmatrix} \frac{15}{14}\nu_3 & -\frac{1}{7}\nu_3+\frac{2}{7}\\ -\frac{5}{8}\nu_3+\frac{5}{2} & -\frac{3}{4}\nu_3-\frac{1}{2}\end{pmatrix} +\\
&\qquad \varphi(c_2)\big\|_\kappa \begin{pmatrix} -\frac{15}{14}\nu_3-\frac{5}{7} & \frac{1}{7}\nu_3+\frac{2}{7}\\ \frac{5}{8}\nu_3-\frac{5}{2} & \frac{1}{4}\nu_3\end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_{2,0}\varphi(c_0) &= \varphi(c_0)\big\|_\kappa \begin{pmatrix} -\frac{21}{2}\nu_3 & 2\nu_3-4\\ \frac{7}{2}\nu_3-14 & 6\nu_3+4\end{pmatrix} +\\
&\qquad \varphi(c_0)\big\|_\kappa \begin{pmatrix} \frac{7}{2}\nu_3+7 & -2\nu_3-4\\ -\frac{7}{2}\nu_3+14 & -2\nu_3\end{pmatrix}\\
\varepsilon_{2,1}\varphi(c_1) &= \varphi(c_1)\big\|_\kappa \left(\begin{smallmatrix} -\frac{7}{5}\nu_3+\frac{7}{5} & -\frac{8}{5}\\ 0 & 2\nu_3+2\end{smallmatrix}\right)\\
\varepsilon_{2,2}\varphi(c_2) &= 0.
\end{aligned}$$

Thus, the matrix of U_3 will have the form,

$$A = \begin{pmatrix} \varepsilon_{0,0} & \varepsilon_{0,1} & \varepsilon_{0,2}\\ \varepsilon_{1,0} & \varepsilon_{1,1} & \varepsilon_{1,2}\\ \varepsilon_{2,0} & \varepsilon_{2,1} & \varepsilon_{2,2}\end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_{0,1} & \varepsilon_{0,2}\\ \varepsilon_{1,0} & 0 & \varepsilon_{1,2}\\ \varepsilon_{2,0} & \varepsilon_{2,1} & 0\end{pmatrix}$$

noticing that $\varepsilon_{i,i} = 0$ for $i = 1, 2, 3$, which shows that the trace of U_3 is zero. We shall use $\varepsilon_{i,j}$ to denote both the endomorphism $\varepsilon_{i,j}$ as well as its matrix with respect to the topological basis $\mathfrak{E}_1$ defined below; this should not cause any confusion.

Our next aim is to calculate the generating functions of the non-zero $\varepsilon_{i,j}$. To this end, and to simplify the computations, we regard each $\varepsilon_{i,j}$ as being an endomorphism of $\mathcal{A}_3$, as opposed to being an endomorphism from the j-th copy of $\mathcal{A}_3$ in the direct sum in the left-hand side of (2.1.1) to the i-th copy.

Each $(\varepsilon_{k,l}\varphi)(c_l)$ is a sum of things of the shape,

$$\varphi(c_t)\|_\kappa \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$$

and each of these must be treated separately. We evaluate this action on the topological basis $\mathfrak{E}_1 = \{e_r(z) : 0 \leq r \in \mathbb{Z}\}$ of $\mathcal{A}_3$. From the isomorphism (2.1.1), we may assume that φ is chosen such that $\varphi(c_t) = e_r(z)$ for some r. So now it is clear that

$$\varphi(c_t)\|_\kappa \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(e_r\|_\kappa \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(z) = \frac{\kappa(cz+d)}{(cz+d)^2}\left(\frac{az+b}{cz+d}\right)^r$$

and using the results of Appendix A, we can write

$$\frac{\kappa(cz+d)}{(cz+d)^2}\left(\frac{az+b}{cz+d}\right)^r = \sum_{m=0}^{\infty} a_m^{(r)} z^m, \tag{2.1.3}$$

with $a_m^{(r)} \in \mathbb{C}_3$. All that remains unclear is $\kappa(cz+d)$, which may be evaluated as follows: as a notational convenience, we shall switch to using x in place of z. In all cases of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ that we will calculate with, $c \equiv 0 \bmod 9$ and $d \equiv 1 \bmod 9$. Therefore, $cx + d = 1 + 9(c'x + d')$ for some $c', d' \in \mathbb{Z}_3$.

Write $cx + d = 4^\rho$, where $\rho = \frac{\log_3(cx+d)}{\log_3(4)}$. Of course ρ will be a power series in x. It is clear that $\frac{\log_3(cx+d)}{\log_3(4)}$ really does converge since $cx + d = 1 + 9(c'x + d')$ and $\log_3(1 + X)$ converges for $|X|_3 < 1$. Then,

$$4^\rho = \sum_{h=0}^{\infty} 3^h \binom{\rho}{h} = 1 + 3\binom{\rho}{1} + 3^2\binom{\rho}{2} + \ldots$$

where by $\binom{\rho}{h}$ we mean $\frac{1}{h!}\prod_{k=0}^{h-1}(\rho - k)$.

Then $\kappa(cx + d) = \kappa(4^\rho) = \kappa(4)^\rho = 4^{t\rho} = (4^\rho)^t = (cx + d)^t$. Lastly, we write $(cx + d)^t = \exp_3(t \log(cx + d))$. A simple calculation shows that $t \log(cx + d)$ is in the radius of convergence of $\exp_3$ so that $\exp_3(t \log(cx + d))$ converges to an element of $\mathcal{O}_3[[x]]$. Thus we may obtain $a_m^{(r)}$.

2.6 Proposition. *The generating function of the operator* $\|_\kappa \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is given by*

$$\frac{\kappa(cx+d)}{(cx+d)(cx+d-axy-by)}.$$

Proof. $\|_\kappa \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has matrix $(a_m^{(r)})_{0\leq m, r\in\mathbb{Z}}$, with rows indexed by m. The generating function of $\|_\kappa \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is simply $\sum_{0\leq r,m\in\mathbb{Z}} a_m^{(r)} x^m y^r$, and rearranging this formal sum we obtain,

$$\begin{aligned}
\sum_{0\leq r,m\in\mathbb{Z}} a_m^{(r)} x^m y^r &= \sum_{r=0}^{\infty} y^r \sum_{m=0}^{\infty} a_m^{(r)} x^m \\
&= \sum_{r=0}^{\infty} y^r \frac{\kappa(cx+d)}{(cx+d)^2}\left(\frac{ax+b}{cx+d}\right)^r \quad \text{using (2.1.3)} \\
&= \frac{\kappa(cx+d)}{(cx+d)^2}\sum_{r=0}^{\infty} y^r \left(\frac{ax+b}{cx+d}\right)^r \\
&= \frac{\kappa(cx+d)}{(cx+d)^2}\frac{1}{1 - y\left(\frac{ax+b}{cx+d}\right)} \\
&= \frac{\kappa(cx+d)}{(cx+d)(cx+d-axy-by)}.
\end{aligned}$$

■

It is now simply a matter of writing down the rational functions of each $\varepsilon_{k,l}$.

$$\varepsilon_{0,1} : h_{0,1}(x,y) = \frac{\kappa\left(\left(-\frac{1}{4}\nu_3+1\right)x-\frac{3}{4}\nu_3-\frac{1}{2}\right)\left(\left(-\frac{1}{4}\nu_3+1\right)x-\frac{3}{4}\nu_3-\frac{1}{2}\right)^{-1}}{\left(-\frac{1}{4}\nu_3+1\right)x-\frac{3}{4}\nu_3-\frac{1}{2}-\frac{3}{10}\nu_3xy-\left(-\frac{1}{10}\nu_3+\frac{1}{5}\right)y} + \frac{\kappa\left(\left(\frac{1}{4}\nu_3-1\right)x+\frac{1}{4}\nu_3\right)\left(\left(\frac{1}{4}\nu_3-1\right)x+\frac{1}{4}\nu_3\right)^{-1}}{\left(\frac{1}{4}\nu_3-1\right)x+\frac{1}{4}\nu_3+\left(\frac{1}{10}\nu_3+\frac{1}{5}\right)xy-\left(\frac{1}{10}\nu_3+\frac{1}{5}\right)y} \tag{2.1.4}$$

$$\varepsilon_{0,2} : h_{0,2}(x,y) = \frac{\kappa\left(-\frac{1}{4}\nu_3-\frac{1}{4}\right)\left(-\frac{1}{4}\nu_3-\frac{1}{4}\right)^{-1}}{-\frac{1}{4}\nu_3-\frac{1}{4}-\left(\frac{1}{7}\nu_3-\frac{1}{7}\right)xy-\frac{2}{7}y} \tag{2.1.5}$$

$$\varepsilon_{1,0} : h_{1,0}(x,y) = \frac{\kappa\left(2\nu_3+2\right)\left(2\nu_3+2\right)^{-1}}{2\nu_3+2-\left(-5\nu_3+5\right)xy-4y} \tag{2.1.6}$$

$$\varepsilon_{1,2} : h_{1,2}(x,y) = \frac{\kappa\left(\frac{5}{8}\left(4-\nu_3\right)x-\frac{1}{4}\left(3\nu_3+2\right)\right)\left(\frac{5}{8}\left(4-\nu_3\right)x-\frac{1}{4}\left(3\nu_3+2\right)\right)^{-1}}{\frac{5}{8}\left(4-\nu_3\right)x-\frac{1}{4}\left(3\nu_3+2\right)-\frac{15}{14}\nu_3xy+\frac{1}{7}\left(\nu_3-2\right)y} + \frac{\kappa\left(\frac{5}{8}\left(\nu_3-4\right)x+\frac{1}{4}\nu_3\right)\left(\frac{5}{8}\left(\nu_3-4\right)x+\frac{1}{4}\nu_3\right)^{-1}}{\frac{5}{8}\left(\nu_3-4\right)x+\frac{1}{4}\nu_3+\frac{5}{14}\left(\nu_3+2\right)xy-\frac{1}{7}\left(\nu_3+2\right)y} \tag{2.1.7}$$

$$\varepsilon_{2,0} : h_{2,0}(x,y) = \frac{\kappa\left(\frac{7}{2}\left(\nu_3-4\right)x+6\nu_3+4\right)\left(\frac{7}{2}\left(\nu_3-4\right)x+6\nu_3+4\right)^{-1}}{\frac{7}{2}\left(\nu_3-4\right)x+6\nu_3+4+\frac{21}{2}\nu_3xy+\left(4-2\nu_3\right)y} + \frac{\kappa\left(\frac{7}{2}\left(4-\nu_3\right)x-2\nu_3\right)\left(\frac{7}{2}\left(4-\nu_3\right)x-2\nu_3\right)^{-1}}{\frac{7}{2}\left(4-\nu_3\right)x-2\nu_3-\frac{7}{2}\left(\nu_3+2\right)xy+\left(2\nu_3+4\right)y} \tag{2.1.8}$$

$$\varepsilon_{2,1} : h_{2,1}(x,y) = \frac{\kappa\left(2\nu_3+2\right)\left(2\nu_3+2\right)^{-1}}{2\nu_3+2-\frac{7}{5}\left(\nu_3-1\right)xy-\frac{8}{5}y} \tag{2.1.9}$$

For a given, fixed κ, we may now use either the generating functions of each $\varepsilon_{k,l}$ or the techniques outlined in Appendix A to obtain a formula for the (m,r)-th entry of the matrix with respect to the topological basis $\mathfrak{C}_1$. Hence we obtain the matrix of U_3 with respect to the topological basis $\mathfrak{B}_1$.

2.7 Lemma. *Every non-zero $\varepsilon_{k,l}$ is compact.*

Proof. Note that by definition, $h_{k,l}(x,y) \in \mathcal{O}_3[[x,y]]$. By Corollary 1.10, it suffices to prove that every entry in $\mathrm{D}\left(\frac{1}{3}\right)\varepsilon_{k,l}$ is in $\mathcal{O}_3$. Equivalently, we need to show that $h_{k,l}\left(\frac{1}{3}x,y\right)$ lies in $\mathcal{O}_3[[x,y]]$. It is simply a case of checking that every coefficient of x is divisble by 3 in $\mathcal{O}_3$. ■

Unfortunately the topological basis $\mathfrak{B}_1$ does not suffice to prove anything about the slopes of the matrix of U_3. As an example of this, we briefly consider the case $\kappa(x) = x^3$. (See Section B.3 for the details.) Here, the slopes of U_3 were found to be

$$\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, 3, \frac{7}{2}, \frac{7}{2}, \frac{9}{2}, \frac{9}{2}, 5, \frac{11}{2}, \frac{11}{2}, \ldots$$

Our goal is to change bases to a topological basis which separates out the integers from the halves of odd integers. More precisely, we will introduce a new topological basis and conjugate the matrix of U_3 to obtain its matrix with respect to this new topological basis. The matrix of U_3 with respect to this new topological basis will be of a particularly straightforward form and it is then almost elementary to prove something about the slopes of U_3.

Firstly we introduce an operator, W: define $\mu \in D_f^\times$ as follows.

$$\mu_q = \begin{cases} 1 \text{ if } q \neq 3 \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right) \text{ if } q = 3 \end{cases}$$

The operator W is defined by

$$W := [U\mu U]\,. \tag{2.1.10}$$

2.8 Remark. 1. W, in fact, corresponds directly, via the Jacquet-Langlands correspondence, to the diamond operator $\langle 4 \rangle$ from the classical theory.

2. W^3 is the identity map on $\mathcal{A}_3^3$.

As for U_3 it is necessary to decompose $U\mu U$ into a disjoint union of single cosets. Again, it is enough to consider everything at 3 since μ is trivial at all places different from 3. It is trivial to prove,

2.9 Lemma. *Let G be as in Lemma 2.3. Then $G\left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)G = G\left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)$.* ■

Now we evaluate $W\varphi$ on c_r, $0 \leq r \leq 2$; following the definitions through, we have

$$\begin{aligned} (W\varphi)(c_r) &= (\varphi|_\kappa \mu)(c_r) \\ &= \varphi\left(c_r\mu^{-1}\right)\|_\kappa \mu_3 \\ &= \sum_{i=0}^{2} \delta_{r,i}\varphi(c_i) \end{aligned}$$

where we regard the $\delta_{r,i}$'s as endomorphisms just as in the case of the $\varepsilon_{k,l}$'s. From Section B.2, we find that

$$c_0\mu^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)^{-1} = (-1)c_1 \begin{pmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{1}{8} \end{pmatrix} \tag{2.1.11}$$

$$c_1\mu^{-1} = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)^{-1} = (-1)c_2 \begin{pmatrix} -\frac{5}{7} & 0 \\ 0 & -\frac{1}{8} \end{pmatrix} \tag{2.1.12}$$

$$c_2\mu^{-1} = \left(\begin{smallmatrix} 7 & 0 \\ 0 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)^{-1} = c_0\left(\begin{smallmatrix} 7 & 0 \\ 0 & 1 \end{smallmatrix}\right) \tag{2.1.13}$$

Recall our convention to identify an element with its 3-part if it is trivial at all other places. Continuing, we obtain

$$\begin{aligned}
(W\varphi)(c_0) &= \varphi(c_1)\|_\kappa \left(\begin{smallmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right) = \delta_{0,1}\varphi(c_1) \\
(W\varphi)(c_1) &= \varphi(c_2)\|_\kappa \left(\begin{smallmatrix} -\frac{5}{7} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right) = \delta_{1,2}\varphi(c_2) \\
(W\varphi)(c_2) &= \varphi(c_0)\|_\kappa \left(\begin{smallmatrix} 7 & 0 \\ 0 & 4 \end{smallmatrix}\right) = \delta_{2,0}\varphi(c_0)
\end{aligned}$$

so W has matrix

$$W = \begin{pmatrix} 0 & \delta_{0,1} & 0 \\ 0 & 0 & \delta_{1,2} \\ \delta_{2,0} & 0 & 0 \end{pmatrix}$$

Evaluating each $\delta_{k,l}$ on the elements of the topological basis $\mathfrak{C}_1$, regarding each as an endomorphism of $\mathcal{A}_3$, we find (writing $\delta_{i,j}$ for both the endomorphism and its matrix with respect to $\mathfrak{C}_1$):

$$\begin{aligned}
\delta_{0,1} &= 4\kappa\left(-\tfrac{1}{2}\right)\mathrm{D}\left(\tfrac{2}{5}\right) \\
\delta_{2,0} &= 4\kappa\left(-\tfrac{1}{2}\right)\mathrm{D}\left(\tfrac{10}{7}\right) \\
\delta_{0,1} &= \tfrac{1}{16}\kappa\left(4\right)\mathrm{D}\left(\tfrac{7}{4}\right)
\end{aligned}$$

recalling the notation (1.1.2). Explicilty, the matrix of W with respect to $\mathfrak{B}_1$ is

$$\begin{pmatrix} 0 & 4\kappa\left(-\frac{1}{2}\right)\mathrm{D}\left(\frac{2}{5}\right) & 0 \\ 0 & 0 & 4\kappa\left(-\frac{1}{2}\right)\mathrm{D}\left(\frac{10}{7}\right) \\ \frac{1}{16}\kappa\left(4\right)\mathrm{D}\left(\frac{7}{4}\right) & 0 & 0 \end{pmatrix}$$

Since W^3 is the identity map on $\mathcal{A}_3^3$, it follows from linear algebra that $\mathcal{A}_3^3$ splits into three subspaces associated to W, namely the eigenspaces. Clearly W has minimal polynomial $F(X) = X^2 + X + 1$. Let $\omega = \frac{-1+\sqrt{-3}}{2} \in \mathbb{C}_3$ be a root of $F(X)$. Let the eigenspaces of W be $K_0 = \ker(W - I)$, $K_1 = \ker(W - \omega I)$ and $K_2 = \ker(W - \omega^2 I)$. Now W commutes with U_3, and so U_3 stabilizes each K_t. Our goal now is to choose bases for each K_t. It may be readily verified that

- $b_r^{(0)}(z) = \begin{pmatrix} 16\kappa\left(\frac{1}{4}\right)e_r\left(\frac{4}{7}z\right) \\ 4\kappa\left(-\frac{1}{2}\right)e_r\left(\frac{10}{7}z\right) \\ e_r(z) \end{pmatrix}$, $0 \le r \in \mathbb{Z}$, is a topological basis for K_0
- $b_r^{(1)}(z) = \begin{pmatrix} 16\kappa\left(\frac{1}{4}\right)\omega e_r\left(\frac{4}{7}z\right) \\ 4\kappa\left(-\frac{1}{2}\right)\omega^2 e_r\left(\frac{10}{7}z\right) \\ e_r(z) \end{pmatrix}$, $0 \le r \in \mathbb{Z}$, is a topological basis for K_1

- $b_r^{(2)}(z) = \begin{pmatrix} 16\kappa\left(\frac{1}{4}\right)\omega^2 e_r\left(\frac{4}{7}z\right) \\ 4\kappa\left(-\frac{1}{2}\right)\omega e_r\left(\frac{10}{7}z\right) \\ e_r(z) \end{pmatrix}$, $0 \le r \in \mathbb{Z}$, is a topological basis for K_2

Let $\mathfrak{B}_2$ denote the topological basis $\left\{b_r^{(0)}(z), b_s^{(1)}(z), b_t^{(2)}(z) : 0 \le r, s, t \in \mathbb{Z}\right\}$ of $\mathcal{A}_3^3$.

It is clear the change of basis matrix is given by

$$B = \begin{pmatrix} 16\kappa\left(\frac{1}{4}\right)\mathrm{D}\left(\frac{4}{7}\right) & 16\omega\kappa\left(\frac{1}{4}\right)\mathrm{D}\left(\frac{4}{7}\right) & 16\omega^2\kappa\left(\frac{1}{4}\right)\mathrm{D}\left(\frac{4}{7}\right) \\ 4\kappa\left(-\frac{1}{2}\right)\mathrm{D}\left(\frac{10}{7}\right) & 4\omega^2\kappa\left(-\frac{1}{2}\right)\mathrm{D}\left(\frac{10}{7}\right) & 4\omega\kappa\left(-\frac{1}{2}\right)\mathrm{D}\left(\frac{10}{7}\right) \\ \mathrm{D}(1) & \mathrm{D}(1) & \mathrm{D}(1) \end{pmatrix}$$

and moreover, B is invertible. We find that,

$$3B^{-1} = \begin{pmatrix} \frac{1}{16}\kappa(4)\mathrm{D}\left(\frac{7}{4}\right) & \frac{1}{4}\kappa(-2)\mathrm{D}\left(\frac{7}{10}\right) & \mathrm{D}(1) \\ \frac{1}{16}\omega^2\kappa(4)\mathrm{D}\left(\frac{7}{4}\right) & \frac{1}{4}\omega\kappa(-2)\mathrm{D}\left(\frac{7}{10}\right) & \mathrm{D}(1) \\ \frac{1}{16}\omega\kappa(4)\mathrm{D}\left(\frac{7}{4}\right) & \frac{1}{4}\omega^2\kappa(-2)\mathrm{D}\left(\frac{7}{10}\right) & \mathrm{D}(1) \end{pmatrix}$$

We have,

2.10 Lemma.

$$B^{-1}AB = \begin{pmatrix} \begin{matrix}\frac{1}{16}\kappa(4)\mathrm{D}\left(\frac{7}{4}\right)\varepsilon_{0,2}+ \\ \frac{1}{4}\kappa(-2)\mathrm{D}\left(\frac{7}{10}\right)\varepsilon_{1,2}\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}\frac{1}{16}\omega^2\kappa(4)\mathrm{D}\left(\frac{7}{4}\right)\varepsilon_{0,2}+ \\ \frac{1}{4}\omega\kappa(-2)\mathrm{D}\left(\frac{7}{10}\right)\varepsilon_{1,2}\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}\frac{1}{16}\omega\kappa(4)\mathrm{D}\left(\frac{7}{4}\right)\varepsilon_{0,2}+ \\ \frac{1}{4}\omega^2\kappa(-2)\mathrm{D}\left(\frac{7}{10}\right)\varepsilon_{1,2}\end{matrix} \end{pmatrix} \tag{2.1.14}$$

■

The proof of Lemma 2.1.14 relies heavily on a large amount of cancellation in calculation of $B^{-1}AB$; since U_3 and W commute, we can easily verify the following equalities from which all the cancellation is evident.

2.11 Lemma. *The following equalities hold:*

$$\tfrac{1}{16}\kappa(4)h_{0,2}\left(\tfrac{7}{4}x, y\right) = 4\kappa\left(-\tfrac{1}{2}\right)h_{2,1}\left(x, \tfrac{10}{7}y\right) = 4\kappa\left(-\tfrac{1}{2}\right)h_{1,0}\left(\tfrac{7}{10}x, \tfrac{4}{7}y\right)$$

and

$$16\kappa\left(\tfrac{1}{4}\right)h_{2,0}\left(x, \tfrac{4}{7}y\right) = \tfrac{1}{4}\kappa(-2)h_{1,2}\left(\tfrac{7}{10}x, y\right) = \tfrac{1}{4}\kappa(-2)h_{0,1}\left(\tfrac{7}{4}x, \tfrac{10}{7}y\right).$$

■

Or, bringing Proposition 1.3 into play, in terms of ε's we have,

$$\tfrac{1}{16}\kappa(4)\,\mathrm{D}\left(\tfrac{7}{4}\right)\varepsilon_{0,2} = 4\kappa\left(-\tfrac{1}{2}\right)\varepsilon_{2,1}\,\mathrm{D}\left(\tfrac{10}{7}\right) = 4\kappa\left(-\tfrac{1}{2}\right)\mathrm{D}\left(\tfrac{7}{10}\right)\varepsilon_{1,0}\,\mathrm{D}\left(\tfrac{4}{7}\right)$$

and

$$16\kappa\left(\tfrac{1}{4}\right)\varepsilon_{2,0}\,\mathrm{D}\left(\tfrac{4}{7}\right)=\tfrac{1}{4}\kappa(-2)\,\mathrm{D}\left(\tfrac{7}{10}\right)\varepsilon_{1,2}=\tfrac{1}{4}\kappa(-2)\,\mathrm{D}\left(\tfrac{7}{4}\right)\varepsilon_{0,1}\,\mathrm{D}\left(\tfrac{10}{7}\right).$$

Set $M_{2,2}=\frac{1}{16}\omega^2\kappa(4)\,\mathrm{D}\left(\frac{7}{4}\right)\varepsilon_{0,2}+\frac{1}{4}\omega\kappa(-2)\,\mathrm{D}\left(\frac{7}{10}\right)\varepsilon_{1,2}$. The rational function of $M_{2,2}$ is $H_{2,2}(x,y)=\frac{1}{16}\omega^2\kappa(4)h_{0,2}\left(\frac{7}{4}x,y\right)+\frac{1}{4}\omega\kappa(-2)h_{1,2}\left(\frac{7}{10}x,y\right)$. We shall prove that the n-th slope of $M_{2,2}$, for $n\in\mathbb{N}$, is $n-\frac{1}{2}$. The strategy to do this is to study $H_{2,2}(x,y)$ in order to verify the following.

2.12 Theorem. *Suppose that $M=(M_{i,j})_{0\le i,j\in\mathbb{Z}}$ is an infinite matrix over $\mathcal{O}_3$ and is compact. Define $N=(N_{i,j})_{0\le i,j\in\mathbb{Z}}$ as follows,*

$$N_{i,j}:=\frac{1}{3^i}M_{i,j}.$$

If $N_{i,j}\in\mathcal{O}_3$ for all $0\le i,j\in\mathbb{Z}$ and $\det(N_{i,j})_{0\le i,j\le n}$ is a unit for all $0\le n\in\mathbb{Z}$, then the slopes of M are $0,1,2,3,4,5,\ldots$.

Proof. To prove this result, we use the explicit formula for c_m as given in (1.2.3). Recall that if S is a finite subset of $I=\mathbb{Z}_{\ge 0}$, and σ is a permutation of S we write

$$M_{S,\sigma}=\prod_{i\in S}M_{\sigma(i),i}\quad\text{and}\quad c_S=\sum_{\sigma\in\mathrm{Sym}(S)}\mathrm{sgn}(\sigma)M_{S,\sigma}$$

Then,

$$c_m=(-1)^m\sum_{\substack{S\subseteq I\\|S|=m}}c_S. \tag{2.1.15}$$

Let $m\in\mathbb{N}$ and define $S_m=\{0,1,\ldots,m-1\}$ so that $|S_m|=m$. We shall prove the following:

- $v_3\left(c_{S_m}\right)=\frac{m(m-1)}{2}$.
- $v_3\left(M_{S,\sigma}\right)>\frac{m(m-1)}{2}$ for all $S\neq S_m$ such that $|S|=m$.

For the first point, observe that c_{S_m} is in fact the top left $m\times m$ minor of M. Write $\mathrm{D}_m(\alpha)$ to denote the diagonal matrix $\mathrm{diag}\left(1,\alpha,\ldots,\alpha^{m-1}\right)$. The condition that $\det(N_{i,j})_{0\le i,j\le n}$ is a unit for all n is exactly the same as saying that $\det\left(\mathrm{D}_m\left(\frac{1}{3}\right)\right)c_{S_m}$ is a unit for all m, or in other words, that the valuation of $\det\left(\mathrm{D}_m\left(\frac{1}{3}\right)\right)c_{S_m}$ is zero. I.e.,

$$v_3\left(c_{S_m}\right)=v_3\left(\det\left(\mathrm{D}_m(3)\right)\right)$$

But $\det\left(\mathrm{D}_m(3)\right)=1\cdot 3\cdot 3^2\cdot\cdots\cdot 3^{m-1}=3^{\frac{m(m-1)}{2}}$. Thus, we are done.

For the second point, we proceed by induction on $|S|$: we assume the result holds for all S such that $S_m\neq S\subset I$ and $|S|=m$. Let $S'=\{i_0<i_1<\ldots<i_m\}$ be a subset of I

consisting of $m+1$ elements and different from S_{m+1}. Consider $v_3\left(M_{S',\sigma'}\right)$ where σ' is a permutation of S'. We have

$$\begin{aligned} v_3\left(M_{S',\sigma'}\right) &= \sum_{i\in S'} v_3\left(M_{\sigma'(i),i}\right) \\ &\geq \sum_{i\in S'} \sigma'(i) \text{ because of the condition } N_{i,j}\in\mathcal{O}_3 \\ &= \sum_{i\in S'} i. \end{aligned}$$

As $S'\neq S_{m+1}$, there is an $s'\in S'$ such that $s'>m+1$. Since $S'=\{s'\}\cup(S'-\{s'\})$, and $S'-\{s'\}$ consists of m elements, we have that $\sum_{i\in S'-\{s'\}} i \geq \frac{m(m-1)}{2}$ — this is simply the fact that the smallest possible value for the sum of m non-negative distinct integers is $\frac{m(m-1)}{2}$. Hence,

$$v_3\left(M_{S',\sigma'}\right) > m+1+\frac{m(m-1)}{2} = 1+\frac{m(m+1)}{2}$$

This completes the induction, except for the base case: for $m=1$ we need to show that for all $S\neq S_1=\{0\}$ and $|S|=1$ we have that $v_3(M_{S,\sigma})>0$. The conditions on S mean that $S=\{s\}$ and $s\in\mathbb{N}$. Moreover there is only one σ — the identity permutation. It follows that $M_{S,\sigma}=M_{s,s}$ and by assumption $v_3(M_{s,s})\geq s>0=\frac{0(0-1)}{2}$ and we are done.

It now follows that $v_3\left(c_S\right)>\frac{m(m-1)}{2}$ for all $S\neq S_m$ such that $|S|=m$ since it is the sum of things all of whose valuations are greater than $1+\frac{m(m+1)}{2}$.

Hence c_m has valuation equal to $\frac{m(m-1)}{2}$ since it is the sum of one thing that has this valuation and other things that have strictly greater valuation. The set of "valuation coordinates", i.e. the set $P=\{(0,0),(1,v_p(a_1)),(2,v_p(a_2)),\ldots\}$ is then,

$$\{(0,0),(1,0),(2,1),(3,2),(4,6),\ldots\}$$

and since these all lie on the parabola $\frac{1}{2}x(x-1)$ they are on the lower boundary of a convex polygon. Hence these are the vertices of the Newton polygon of M. The sequence of gradients is $0,1,2,3,4,5\ldots$. I.e. the slopes are $0,1,2,3,4,5\ldots$. ■

In view of Proposition 1.3, the condition of the theorem that $N_{i,j}$ be in $\mathcal{O}_3$ is equivalent to the statement $H_M\left(\frac{1}{3}x,y\right)\in\mathcal{O}_3[[x,y]]$ where $H_M(x,y)$ is the generating function of M; or that the coefficient of x^ky^l is in $3^k\mathcal{O}_3$.

Recall that $\mathfrak{m}_3$ is the maximal ideal of $\mathcal{O}_3$; thus the condition that $\det(N_{i,j})_{0\leq i,j\leq n}$ is a unit for all $0\leq n\in\mathbb{Z}$ is equivalent to $\det(N_{i,j})_{0\leq i,j\leq n}$ being a unit modulo $\mathfrak{m}_3$. In other words, the top left $(n+1)\times(n+1)$ minors of $\mathrm{D}\left(\frac{1}{3}\right)M$ are units for all $0\leq n\in\mathbb{Z}$.

Define $M'_{2,2}=\frac{1}{\omega(2\omega+1)}\,\mathrm{D}\left(\frac{1}{2\omega+1}\right)M_{2,2}\,\mathrm{D}(2\omega+1)$ — notice that this is a multiple of a conjugate the matrix $M_{2,2}$. In particular, $M_{2,2}$ and $\mathrm{D}\left(\frac{1}{2\omega+1}\right)M_{2,2}\,\mathrm{D}(2\omega+1)$ will have the same slopes because they have the same characteristic power series. To see that they have

have the same charateristic power series, we invoke Proposition 1.14: let $u = \mathrm{D}\left(\frac{1}{2\omega+1}\right) M_{2,2}$ and $v = \mathrm{D}(2\omega+1)$. It is obvious that v is compact. For u, we examine the definition of u:

$$u = \mathrm{D}\left(\frac{1}{2\omega+1}\right)\left(\tfrac{1}{16}\omega^2\kappa(4)\,\mathrm{D}\left(\tfrac{7}{4}\right)\varepsilon_{0,2} + \tfrac{1}{4}\omega\kappa(-2)\,\mathrm{D}\left(\tfrac{7}{10}\right)\varepsilon_{1,2}\right)$$

Since infinite diagonal matrices commute, we may write

$$u = \tfrac{1}{16}\omega^2\kappa(4)\,\mathrm{D}\left(\tfrac{7}{4}\right)\mathrm{D}\left(\frac{1}{2\omega+1}\right)\varepsilon_{0,2} + \tfrac{1}{4}\omega\kappa(-2)\,\mathrm{D}\left(\tfrac{7}{10}\right)\mathrm{D}\left(\frac{1}{2\omega+1}\right)\varepsilon_{1,2}$$

and compactness of u now follows from the proof of Lemma 2.7, since u is the sum of two compact operators.

As the slopes are merely the valuations of the inverses of the eigenvalues of $M_{2,2}$, the presence of the multipicative factor of $\frac{1}{\omega(2\omega+1)}$ in the definition of $M'_{2,2}$ will simply have the effect of adding $v_3(\omega(2\omega+1)) = \frac{1}{2}$ to each slope.

The rational function of $M'_{2,2}$ is then $H'_{2,2}(x,y) := \frac{1}{\omega(2\omega+1)} H_{2,2}\left(\frac{x}{2\omega+1}, y(2\omega+1)\right)$.

2.13 Lemma. *$H'_{2,2}\left(\frac{1}{3}x, y\right)$ is an element of $\mathcal{O}_3[[x,y]]$.*

Proof. This is a highly unilluminating check on the coefficients. The result will follow from our proof of Theorem 2.14 below. ∎

The final step is to show that $\det(N_{i,j})_{0\le i,j\le n}$ is a unit modulo $\mathfrak{m}_3$ for all $0 \le n \in \mathbb{Z}$, where $N = \mathrm{D}\left(\frac{1}{3}\right) M'_{2,2}$. To do this, we study the rational function of N; this is simply $H'_{2,2}\left(\frac{1}{3}x, y\right)$.

2.14 Theorem.

$$H'_{2,2}\left(\tfrac{1}{3}x, y\right) \equiv \frac{-1}{1-xy} \bmod \mathfrak{m}_3.$$

2.15 Corollary. $N \equiv -\mathrm{D}(1) \bmod \mathfrak{m}_3$.

Proof. This is merely a restatement of Theorem 2.14 in terms of matrices, via Proposition 1.3. ∎

Proof of Theorem 2.14. The only way to analyse this rational function is to look explicitly at the numerator and denominator. The numerator is the sum of

$$\big(- 2420208\omega y^2x^4 - 26622288\omega\nu_3y^2x^4 + 14000231\omega x^4 - 46118408\omega\nu_3x^4 + 52706752\omega yx^4 + 8000132\omega\nu_3yx^4 + 16000264\nu_3yx^4 + 105413504yx^4 + 116169984\omega y^2x^3 - 66382848\omega\nu_3y^2x^3 - 132765696\nu_3y^2x^3 + 232339968y^2x^3 + 316240512\omega x^3 + 48000792\omega\nu_3x^3 + 96001584\nu_3x^3 - 38723328\omega yx^3 - 425956608\omega\nu_3yx^3 + 632481024x^3 - 768144384\omega y^2x^2 - 128024064\omega\nu_3y^2x^2 + 435637440\omega x^2 - 1089093600\omega\nu_3x^2 + 647232768\omega yx^2 - 273829248\omega\nu_3yx^2 - 547658496\nu_3yx^2 + 1294465536yx^2 - 146313216\omega y^2x - 438939648\omega\nu_3y^2x - 877879296\nu_3y^2x - 292626432y^2x + 945955584\omega x + 323616384\omega\nu_3x + 647232768\nu_3x - 1450939392\omega yx - 341397504\omega\nu_3yx + 1891911168x - 564350976\omega y^2 + 188116992\omega\nu_3y^2 + 448084224\omega - 384072192\omega\nu_3 - 402361344\omega\nu_3y - 804722688\nu_3y)\big)\kappa(-\nu_3 - 1),$$

and

$$\big(- 49787136\omega y^2x^3 - 99574272\omega\nu_3y^2x^3 - 49787136\nu_3y^2x^3 - 24893568y^2x^3 - 58084992\nu_3yx^3 + 101648736yx^3 - 512096256\nu_3y^2x^2 + 768144384y^2x^2 - 261382464\nu_3x^2 - 1642975488\omega yx^2 - 597445632\omega\nu_3yx^2 - 298722816\nu_3yx^2 - 821487744yx^2 - 130691232x^2 - 3072577536\omega y^2x - 877879296\omega\nu_3y^2x - 438939648\nu_3y^2x - 1536288768y^2x - 2688505344\omega x + 896168448\omega\nu_3x + 448084224\nu_3x - 4096770048\nu_3yx + 896168448yx - 1344252672x - 1755758592\nu_3y^2 - 250822656y^2 - 2560481280\nu_3 - 7461974016\omega y + 877879296\omega\nu_3y + 438939648\nu_3y - 3730987008y - 2176409088\big)\kappa\left(\tfrac{7}{72}(2\omega+1)x(\nu_3-4) - \tfrac{\nu_3}{2}\right)$$

and

$$\big(- 82978560\omega y^2x^3 - 16595712\omega\nu_3y^2x^3 - 8297856\nu_3y^2x^3 - 41489280y^2x^3 - 58084992\nu_3yx^3 + 101648736yx^3 - 256048128\nu_3y^2x^2 - 256048128y^2x^2 - 261382464\nu_3x^2 - 746807040\omega yx^2 - 149361408\omega\nu_3yx^2 - 74680704\nu_3yx^2 - 373403520yx^2 - 130691232x^2 - 146313216\omega y^2x + 731566080\omega\nu_3y^2x + 365783040\nu_3y^2x - 73156608y^2x - 896168448\omega x - 256048128\nu_3yx + 128024064yx - 448084224x - 752467968y^2 - 256048128\nu_3 + 146313216\omega y + 146313216\omega\nu_3y + 73156608\nu_3y + 73156608y + 128024064\big)\kappa\left(-\tfrac{7}{72}(2\omega+1)x(\nu_3-4) + \tfrac{1}{2}(3\nu_3+2)\right)$$

and the denominator is

$$4840416\omega y^3x^5 + 53244576\omega\nu_3y^3x^5 + 26622288\nu_3y^3x^5 + 2420208y^3x^5-$$
$$24000396\nu_3y^2x^5 - 158120256y^2x^5 - 28000462\omega yx^5 + 92236816\omega\nu_3yx^5+$$
$$46118408\nu_3yx^5 - 14000231yx^5 + 298722816\nu_3y^3x^4 - 522764928y^3x^4+$$
$$546967008\omega y^2x^4 + 1311752736\omega\nu_3y^2x^4 + 655876368\nu_3y^2x^4+$$
$$273483504y^2x^4 + 340946802\omega x^4 + 148237740\omega\nu_3x^4 + 74118870\nu_3x^4+$$
$$258357204\nu_3yx^4 - 1567084680yx^4 + 170473401x^4 + 3243276288\omega y^3x^3+$$
$$597445632\omega\nu_3y^3x^3 + 298722816\nu_3y^3x^3 + 1621638144y^3x^3+$$
$$3310844544\nu_3y^2x^3 - 4281693696y^2x^3 + 1753440696\nu_3x^3+$$
$$4124034432\omega yx^3 + 5372861760\omega\nu_3yx^3 + 2686430880\nu_3yx^3+$$
$$2062017216yx^3 - 1524731040x^3 + 4389396480\nu_3y^3x^2 + 2194698240y^3x^2+$$
$$13655900160\omega y^2x^2 + 426746880\omega\nu_3y^2x^2 + 213373440\nu_3y^2x^2+$$
$$6827950080y^2x^2 + 7841473920\omega x^2 + 3920736960\omega\nu_3x^2 + 1960368480\nu_3x^2+$$
$$5974456320\nu_3yx^2 - 10455298560yx^2 + 3920736960x^2 + 6145155072\omega y^3x-$$
$$4389396480\omega\nu_3y^3x - 2194698240\nu_3y^3x + 3072577536y^3x+$$
$$7791178752\nu_3y^2x + 8339853312y^2x + 4704884352\nu_3x + 6785275392\omega yx+$$
$$4352818176\omega\nu_3yx + 2176409088\nu_3yx + 3392637696yx - 6721263360x+$$
$$1289945088\nu_3y^3 + 3224862720y^3 + 5141864448\omega y^2 - 5392687104\omega\nu_3y^2-$$
$$2696343552\nu_3y^2 + 2570932224y^2 + 2176409088\omega + 2560481280\omega\nu_3+$$
$$1280240640\nu_3 + 987614208\nu_3y + 658409472y + 1088204544$$

Notice that even though all of the integers in the coefficients of the numerator and denominator are all of roughly the same order, both lie in $\mathcal{O}_3[x,y]$. We have not reduced modulo anything as yet.

The next step is the evaluation of κ at $-\nu_3-1$, $\frac{7}{72}(2\omega+1)x(\nu_3-4)-\frac{\nu_3}{2}$ and $-\frac{7}{72}(2\omega+1)x(\nu_3-4)+\frac{1}{2}(3\nu_3+2)$. Recall that we fixed $t\in\mathfrak{m}_3$ such that $\kappa(4)=4^t$ and we showed above that $\kappa(cx+d)=(cx+d)^t$ where by $(cx+d)^t$ we mean $\exp_3(t\log_3(cx+d))$.

For the case, $-\nu_3-1$, we have $-\nu_3-1=1+(-2-\nu_3)$ and $-2-\nu_3\equiv 0 \bmod 3$. Thus, $\log_3(-\nu_3-1)=3+9L_0$ with $L_0\in\mathbb{Z}_3^\times$. Then, $v_3(t\log_3(-\nu_3-1))=v_3(t)+1>1$, so that $t\log_3(-\nu_3-1)$ is in the disc of convergence of $\exp_3$. Hence $\exp_3(t(3+9L_0))$ converges to an element $1+(2\omega+1)tL_1$ with $L_1\in\mathcal{O}_3$. This follows from examining the power series expansions of $\exp_3$ and noting that $-3=(2\omega+1)^2$

For the cases of $\frac{7}{72}(2\omega+1)x(\nu_3-4)-\frac{\nu_3}{2}$ and $-\frac{7}{72}(2\omega+1)x(\nu_3-4)+\frac{1}{2}(3\nu_3+2)$, note that each is of the form $cx+d$ and that $v_3(c)=\frac{1}{2}$ and $v_3(d-1)=1$. So $\log_3(cx+d)$ converges to a convergent power series in $\mathcal{O}_3[[x]]$ in which every coefficient has valuation at least $\frac{1}{2}$. Thus, $\log_3(cx+d) \in (2\omega+1)\mathcal{O}_3[[x]]$. Multiplying by t gives an element of $(2\omega+1)t\mathcal{O}_3[[x]]$ and exponentiating gives an element of $1+(2\omega+1)t\mathcal{O}_3[[x]]$. Notice that $t\mathcal{O}_3 \subset \mathfrak{m}_3$ since $v_3(t) > 0$ and therefore $1+(2\omega+1)t\mathcal{O}_3[[x]] \subset 1+\mathfrak{m}_3[[x]]$. Put $\kappa\left(\frac{7}{72}(2\omega+1)x(\nu_3-4) - \frac{\nu_3}{2}\right) = 1+(2\omega+1)tL_2$ and $\kappa\left(-\frac{7}{72}(2\omega+1)x(\nu_3-4)+\frac{1}{2}(3\nu_3+2)\right) = 1+(2\omega+1)tL_3$ with $L_2, L_3 \in \mathcal{O}_3[[x]]$.

The next step is to divide out all common powers of 3 from the numerator and denominators. We make the substitution $\nu_3 = 2695 + 3^{10}S$ for $S \in \mathbb{Z}_3^\times$. We find that the numerator may be written as

$$3^6\big(-(2\omega+1)(x^2y^2+xy+1) + t(2\omega+1)L_1(x^2y^2+xy+1) + t(2\omega+1)L_2(x^2y^2+xy+1) + t(2\omega+1)L_3(x^2y^2+xy+1)\big) + 3^7L_4$$

for some $L_4 \in \mathcal{O}_3[[x,y]]$ and the denominator may be written as

$$3^6\big(-(2\omega+1)x^3y^3 + 2\omega+1\big) + 3^7L_5$$

for some $L_5 \in \mathcal{O}_3[[x,y]]$. Dividing out by $3^6(2\omega+1)$ gives

$$\big(-(x^2y^2+xy+1) + tL_1(x^2y^2+xy+1) + tL_2(x^2y^2+xy+1) + tL_3(x^2y^2+xy+1)\big) - (2\omega+1)L_4$$

for the numerator and

$$\big(1-x^3y^3\big) - (2\omega+1)L_5$$

for the denominator. Modulo $\mathfrak{m}_3$ (recalling that $t, 2\omega+1 \in \mathfrak{m}_3$) the quotient becomes,

$$\frac{-(x^2y^2+xy+1)}{1-x^3y^3}$$

which is formally equal to

$$\frac{-1}{1-xy}.$$

■

2.16 Corollary. *The n-th slope of $M_{2,2}$ is $n - \frac{1}{2}$ for $n \in \mathbb{N}$.*

Proof. Apply Theorem 2.14 with $M = M'_{2,2}$. ■

2.2 Extending the Results

For our main example we focused on maps κ in one particular disc in weight space. It is not too much more effort to prove similar results for the following class of κ's

1. $\kappa(4) = 4^{t+\lambda}$, where $t+\lambda = \frac{\log_3(\kappa(4))}{\log_3(4)}$ for $\lambda \in \{1,2\}$,

2. $v_3(t) > 0$, so that $t \in \mathfrak{m}_3$,

3. and without loss of generality, $\kappa(-1) = 1$.

For each λ, this collection is another disc in weight space.

All that differs in these cases is the evaluation of $(cx+d)^{t+\lambda}$ which we regard as being $(cx+d)^{\lambda}\exp_3(t\log_3(cx+d))$ and the evaluation of the exponential is as above.

Let $M_{3,3} = \frac{1}{16}\omega\kappa(4)\,\mathrm{D}\left(\frac{7}{4}\right)\varepsilon_{0,2} + \frac{1}{4}\omega^2\kappa(-2)\,\mathrm{D}\left(\frac{7}{10}\right)\varepsilon_{1,2}$ (see equation (2.1.14)). A similar analysis to that applied to $M_{2,2}$ will prove that the n-th slope of $M_{3,3}$ is also $n-\frac{1}{2}$ for $n \in \mathbb{N}$. Again we may extend even more to the additional κ's detailed above.

The most obvious way to extend the results is consider other odd primes p. However, as p increases, so does the number of cosets in the decomposition $U\eta_p U$ — see Definitions 1.32 and 1.33. In fact, for $U = U_0(p^n)$ and $U = U_1(p^n)$ there will be exactly p cosets.

We could also consider automorphic forms over a definite quaternion algebra over a totally real field F other than $\mathbb{Q}$ — the so-called Hilbert case, since the automorphic forms correspond to Hilbert Modular Forms. In this case we would have to consider operators $U_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$ of F.

Conclusion

We began introducing the notion of an infinite matrix. Such matrices arise for example as linear maps on Banach space. This led us to reviewing some of the theory of compact operators on Banach spaces, and gave the definition of the characteristic power series — a certain power series associated to each compact operator.

We next introduced the notion of the Newton polygon of a characteristic power series and the slopes of the Newton polygon. The slopes of the Newton polygon give us information about the valuations of the eigenvalues of a compact operator.

Next, we set up the ground work in order to define automorphic forms over a definite quaternion algebra over $\mathbb{Q}$ forming the "adelic quaternion" ring,

$$D_f = D \otimes_{\mathbb{Q}} \mathbb{A}_f.$$

Given a $\mathbb{C}_p$-vector space, A, our space of automorphic forms was defined as a set of maps $D_f^\times \to A$ satisfying certain transformation properties.

The next step was to define a certain kind of operator, called a Hecke operator, on this space. We focused on a particular Hecke operator U_3 and a particular case of the maps κ — the weight of the operator. By studying the rational function of this operator on a certain subspace of the automorphic forms, we proved that the sequence of slopes of U_3 on that subspace is

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{15}{2}, \frac{17}{2}, \frac{19}{2}, \frac{21}{2}, \frac{23}{2}, \ldots$$

namely it is a sequence in arithmetic progression.

Appendix A

Power series calculations

In this appendix, we detail some of the rather elementary, but nonetheless important formulae concerning the coefficients of a formal power series which is the evolution of the quotient or product of two polynomials in one variable.

In other terms, given $M, N \in \mathbb{Z}$, and P, Q, R, S in some field of characteristic zero, K, we wish to determine the coefficients a_m in the formal expansion,

$$(P + Qz)^M (R + Sz)^N = \sum_{m=0}^{\infty} a_m z^m. \tag{A.0.1}$$

To this end, we have:

A.1 Theorem. *For fixed* $0 \leq m \in \mathbb{Z}$, $\frac{d^m}{dz^m}\left((P + Qz)^M (R + Sz)^N\right)$, *the m-th formal derivative of* $(P + Qz)^M (R + Sz)^N$, *is given by*

1. *if* $M > 0$ *and* $N > 0$,

$$\sum_{n=0}^{m} \left(\binom{m}{n} \left(\prod_{q=0}^{m-n-1} (M - q) \right) \left(\prod_{q=0}^{n-1} (N - q) \right) \times Q^{m-n} S^n (P + Qz)^{M-m+n} (R + Sz)^{N-n} \right)$$

2. *if* $M \neq 0$ *and* $N = 0$,

$$\left(\prod_{q=0}^{m-1} (M - q) \right) Q^m (P + Qz)^{M-m}$$

3. *if* $M > 0$ *and* $N < 0$,

$$\sum_{n=0}^{m} \left((-1)^n \binom{m}{n} \left(\prod_{q=0}^{m-n-1} (M - q) \right) \left(\prod_{q=0}^{n-1} (q - N) \right) \times Q^{m-n} S^n (P + Qz)^{M-m+n} (R + Sz)^{N-n} \right)$$

4. *if $M < 0$ and $N < 0$,*

$$\sum_{n=0}^{m}\left((-1)^n\binom{m}{n}\left(\prod_{q=0}^{m-n-1}(q-M)\right)\left(\prod_{q=0}^{n-1}(q-N)\right)\times Q^{m-n}S^n(P+Qz)^{M-m+n}(R+Sz)^{N-n}\right)$$

■

All of these results are easily proved using induction on m.

It is evident that the m-th formal derivative of the right hand side of equation (A.0.1) is

$$m!a_m + z \times (\text{some power series})$$

and evaluating both sides at $z = 0$ gives

A.2 Corollary. *We have,*

1. *if $M > 0$ and $N > 0$,*

$$a_m = \begin{cases} \sum_{n=0}^{m}\left(\binom{M}{m-n}\binom{N}{n}P^{M-m+n}Q^{m-n}R^{N-n}S^n\right), & 0 \le m \le M+N, \\ 0, & m > M+N \end{cases}$$

2. *if $M \neq 0$ and $N = 0$,*

$$a_m = \begin{cases} \binom{M}{m}P^{M-m}Q^m & \text{if } M > 0 \\ (-1)^m\binom{-M}{m}P^{M-m}Q^m & \text{if } M < 0. \end{cases}$$

3. *if $M > 0$ and $N < 0$,*

$$a_m = \sum_{n=\max(0,m-M)}^{m}\left((-1)^n\binom{M}{m-n}\binom{-N}{n}P^{M-m+n}Q^{m-n}R^{N-n}S^n\right)$$

4. *if $M < 0$ and $N < 0$,*

$$a_m = (-1)^m\sum_{n=0}^{m}\left(\binom{-M}{m-n}\binom{-N}{n}P^{M-m+n}Q^{m-n}R^{N-n}S^n\right)$$

■

Appendix B

Program Code Listing

B.1 U_3 Cosets Decompositions

In this section we give the PARI/GP code which was used to determine the cosets to which the $c_i v_t^{-1}$ belong.

PARI does not support quaternions as such. To this end, we represent a quaternion, $a + b\mathfrak{i} + c\mathfrak{j} + d\mathfrak{k}$, in PARI by the expression `[a, b, c, d]`, which is a vector. We write `x0` for ν_3 and `y0` for ξ_3.

We begin by defining functions `th` and `th1` that correspond to our isomorphism θ and its inverse, respectively, as well as declaring the fixed value of `y0`.

```
/* Declare y0 and define isos th: K(i, j) <-> M_2(K) : th1 */


y0 = 1


global(y0)


th(v) = [v[1] + v[2]*x0 + v[4]*y0, v[2]*y0 - v[3] - v[4]*x0; \
v[2]*y0 + v[3] -v[4]*x0, v[1] - v[2]*x0 - v[4]*y0]


th1(m) = [(m[1,1] + m[2,2])/2, \
((m[2,2] - m[1,1])*x0 - (m[1,2] + m[2,1])*y0)/2, \
(m[2,1] - m[1,2])/2, \
((m[2,2] - m[1,1])*y0 + (m[1,2] + m[2,1])*x0)/2]
```

It will be necessary to invert quaternions, `[a, b, c, d]`. The next function we present does exactly this.

```
/* Return the inverse of the "quaternion" [q0, q1, q2, q3] */

quatinv(v) = vv = v; N = sum(ii=1, 4, v[ii]^2); \
for(ii=2,4, vv[ii] *= -1/N); vv[1] /= N; vv
```

The next section of code explicitly performs the computations of Lemma 2.4, i.e. it produces a d^{-1} satisfying the conditions derived in Lemma 2.4.

```
/* Lists d1 in O_D s.t. N(d1) = 3 & d1 = [0, *; 0, *] mod 3: */
d1list=[]

forstep(d0=-2, 2, 1/2, forstep(d1=-2, 2, 1/2, \
forstep(d2=-2, 2, 1/2, forstep(d3=-2, 2, 1/2, \
if(d0^2 + d1^2 + d2^2 + d3^2 == 3, \
mm = subst(th([d0, d1, d2, d3]), x0, sqrt(-2 + O(3^15))); \
if((mm[1, 1]%3 == 0) && (mm[2, 1]%3 == 0), \
d1list = concat(d1list, [[d0, d1, d2, d3]])))))))
```

Here, we compute the units of $\mathcal{O}_D$ and their matrix forms under the isomorphism θ.

```
/* Calculate the units of O_D & matrix versions under th: */

unitlist = []

forstep(a=-2, 2, 1/2, forstep(b=-2, 2, 1/2, \
forstep(cc=-2, 2, 1/2, forstep(d=-2, 2, 1/2, \
if(a^2 + b^2 + cc^2 + d^2 == 1, \
unitlist = concat(unitlist, [[a, b, cc, d]]))))))

matunitlist = vector(length(unitlist), i, th(unitlist[i]))
```

The function `test` determines if a given matrix `mm` has entries in $\mathbb{Z}_3$.

```
/* "test" tests matrices for integral entries: */

test(mm) = MM = mm; temp = matrix(2, 2, i, j, \
valuation(MM[i,j], 3)); mmin = vecmin([vecmin(temp[1,]), \
vecmin(temp[2,])]); mmin >= 0
```

Finally we can begin the decomposition calculations. The next block of code calculates the decomposition of Lemma 2.4:

```
/* Step 1 Factorisation: write c_i v_t^{-1} = du: */

clist = [[1, 0; 0, 1], [5, 0; 0, 2], [7, 0; 0, 4]]

c(i) = clist[i+1]

vt3(t) = [3, 0; 9*t, 1]

fact1list = []

for(t = 0, 2, for(i = 0, 2, for(j=1, length(d1list),\
u = th(d1list[j])*c(i)*(1/vt3(t)); \
uu = subst(u, x0, sqrt(-2 + O(3^15))); \
if(test(uu), fact1list = concat(fact1list,\
[[t, i, j,[quatinv(d1list[j]), u]]]); break(1)))))
```

The output from this code is:

```
/* Output of fact1list: */
for(i=1, length(fact1list), print(fact1list[i]))
[0, 0, 1, [[-1/2, 1/6, 1/6, -1/6], \
[-1/6*x0 - 1/3, -1/2*x0; -1/6*x0 - 1/3, 1/2*x0 - 2]]]
[0, 1, 1, [[-1/2, 1/6, 1/6, -1/6], \
[-5/6*x0 - 5/3, -x0; -5/6*x0 - 5/3, x0 - 4]]]
[0, 2, 1, [[-1/2, 1/6, 1/6, -1/6], \
[-7/6*x0 - 7/3, -2*x0; -7/6*x0 - 7/3, 2*x0 - 8]]]
[1, 0, 1, [[-1/2, 1/6, 1/6, -1/6], \
[4/3*x0 - 1/3, -1/2*x0; -5/3*x0 + 17/3, 1/2*x0 - 2]]]
[1, 1, 1, [[-1/2, 1/6, 1/6, -1/6], \
[13/6*x0 - 5/3, -x0; -23/6*x0 + 31/3, x0 - 4]]]
[1, 2, 1, [[-1/2, 1/6, 1/6, -1/6], \
[29/6*x0 - 7/3, -2*x0; -43/6*x0 + 65/3, 2*x0 - 8]]]
[2, 0, 1, [[-1/2, 1/6, 1/6, -1/6], \
[17/6*x0 - 1/3, -1/2*x0; -19/6*x0 + 35/3, 1/2*x0 - 2]]]
[2, 1, 1, [[-1/2, 1/6, 1/6, -1/6], \
[31/6*x0 - 5/3, -x0; -41/6*x0 + 67/3, x0 - 4]]]
[2, 2, 1, [[-1/2, 1/6, 1/6, -1/6], \
[65/6*x0 - 7/3, -2*x0; -79/6*x0 + 137/3, 2*x0 - 8]]]
```

The form of the output is

```
[t, i, 1, [[a, b, c, d], \
[p, q; r, s]]]
```

meaning that $(c_i v_t^{-1})_3$ has the form $(a + bi + cj + dk)\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$; here we mix quaternion and matrix notation. The output is easier to read in this form.

This piece of code test the factorizations:

```
/* Testing the factorizations: */


for(i=1, length(fact1list), mmmm = fact1list[i]; \
print1(Mod(c(mmmm[2])*(1/vt3(mmmm[1])) - \
th(mmmm[4][1])*mmmm[4][2], x0^2 + 2) "; "))
```

Next, we calculate the decomposition of Lemma 2.5. Here we simply test all the possibilites; there are 3 for `i` and 24 for `alpha`:

```
/* Find alpha & c_i s.t. c_i^{-1} alpha^{-1} u \in U_1(9): */


D = [-1/2, 1/6, 1/6, -1/6]


fl = []


for(i=1, 9, u3 =  fact1list[i][4][2]; for(ii=0, 2, \
for(jj = 1, length(matunitlist), \
v3 = (1/c(ii))*(1/matunitlist[jj])*u3; \
vv33 = subst(v3, x0, sqrt(-2 + O(3^15))); vv33 %= 9; \
if((vv33[2, 1] == 0) && (vv33[2, 2] == 1),  \
fl = concat(fl, [[D, matunitlist[jj], c(ii), v3]])))))
```

The following is modified output of the above and gives the factorizations of the $c_i v_t^{-1}$ in the following order: $c_0 v_0^{-1}$, $c_1 v_0^{-1}$, $c_2 v_0^{-1}$, $c_0 v_1^{-1}$, $c_1 v_1^{-1}$, $c_2 v_1^{-1}$, $c_0 v_2^{-1}$, $c_1 v_2^{-1}$, $c_2 v_2^{-1}$.

```
/* Output: */
for(i=1, length(flX2), print(flX2[i]))
[[-1/3, -1/3, 1/3, 0], [7, 0; 0, 4], \
[1/21*X - 1/21, 2/7; 0, -1/4*X - 1/4]]
[[1/3, 1/3, -1/3, 0], [1, 0; 0, 1], \
[-5/3*X + 5/3, -4; 0, 2*X + 2]]
[[1/3, 1/3, -1/3, 0], [5, 0; 0, 2], \
```

```
[-7/15*X + 7/15, -8/5; 0, 2*X + 2]]
[[-1/6, -1/2, -1/6, -1/6], [5, 0; 0, 2], \
[2/5*X - 3/5, -1/10*X + 1/5; 13/6*X + 11/6, -3/4*X - 1/2]]
[[-1/6, -1/2, -1/6, -1/6], [7, 0; 0, 4], \
[11/14*X - 6/7, -1/7*X + 2/7; 49/24*X + 7/3, -3/4*X - 1/2]]
[[1/6, 1/2, 1/6, 1/6], [1, 0; 0, 1], \
[-19/2*X + 12, 2*X - 4; -101/6*X - 50/3, 6*X + 4]]
[[-1/6, 1/6, 1/2, 1/6], [5, 0; 0, 2], \
[-19/30*X - 19/15, 1/10*X + 1/5; -17/12*X - 1/3, 1/4*X]]
[[-1/6, 1/6, 1/2, 1/6], [7, 0; 0, 4], \
[-41/42*X - 41/21, 1/7*X + 2/7; -31/24*X - 5/6, 1/4*X]]
[[1/6, -1/6, -1/2, -1/6], [1, 0; 0, 1], \
[79/6*X + 79/3, -2*X - 4; 65/6*X + 14/3, -2*X]]
```

The output is of the form

```
[[a, b, c, d], [e, f; g, h], \
[p, q; r, s]]
```

and means that the corresponding $(c_i v_t^{-1})_3$ factorizes as

$$(\mathsf{a} + \mathsf{b}i + \mathsf{c}j + \mathsf{d}\ell) \begin{pmatrix} \mathtt{e} & \mathtt{f} \\ \mathtt{g} & \mathtt{h} \end{pmatrix} \begin{pmatrix} \mathtt{p} & \mathtt{q} \\ \mathtt{r} & \mathtt{s} \end{pmatrix}.$$

B.2 W Cosets Decompositions

In this section we present the code for the factorizations in (2.1.11). We note at once that $c_i\mu^{-1} \in U_0(1)$ for $i = 0, 1, 2$, so that all we require is the last step in the factorization.

All the set-up for these calculations is the same as in Section B.1, and so we omit it.

```
/* Step 2 Factorization: Find alpha \in O_D^\times, c_i (i \in {0, 1, 2})
s.t. c_i^{-1} alpha^{-1} c_r \eta^{-1} \in U_1(9): */


clist = [[1, 0; 0, 1], [5, 0; 0, 2], [7, 0; 0, 4]]


c(i) = clist[i+1]


cre3(r) = c(r)*[1, 0; 0, 1/4]


fl = []
```

```
for(i = 0, 2, for(jj = 1, length(unitlist), for(r = 0, 2, \
v3 = (1/c(i))*(1/th(unitlist[jj]))*cre3(r); \
vv3 = subst(v3, x0, sqrt(-2 + O(3^15))); \
if(test(vv3) && (vv3[2, 1]%9 == 0) && (vv3[2, 2]%9 == 1), \
fl = concat(fl, [[jj, i, r, [cre3(r)], [unitlist[jj], c(i), v3]]])))))
```

This section of code simply tests the possibilities $c_i^{-1}\alpha^{-1}c_r\eta^{-1}$ for fixed $r \in \{0,1,2\}$, and varying $i \in \{0,1,2\}$ and varying $\alpha \in \mathcal{O}_D^\times$ and determines which lies in $U_1(9)$; thus we obtain the factorization. The output from this code is as follows:

```
/* Output: */
for(i=1, length(fl), print(fl[i]))
[24, 0, 2, [[7, 0; 0, 1]], [[1, 0, 0, 0], \
[1, 0; 0, 1], [7, 0; 0, 1]]]
[1, 1, 0, [[1, 0; 0, 1/4]], [[-1, 0, 0, 0], \
[5, 0; 0, 2], [-1/5, 0; 0, -1/8]]]
[1, 2, 1, [[5, 0; 0, 1/2]], [[-1, 0, 0, 0], \
[7, 0; 0, 4], [-5/7, 0; 0, -1/8]]]
```

The output is in the form

```
[a, b, c, [[a11, a12; a21, a22]], [[v1, v2, v3, v4], \
[n11, n12; n21, n22], [u11, u12; u21, u22]]]
```

indicating that the matrix

$$\begin{pmatrix} \mathtt{a11} & \mathtt{a12} \\ \mathtt{a21} & \mathtt{a22} \end{pmatrix}$$

factorizes as

$$(\mathtt{v1} + \mathtt{v2}\mathfrak{i} + \mathtt{v3}\mathfrak{j} + \mathtt{v4}\mathfrak{k}) \begin{pmatrix} \mathtt{n11} & \mathtt{n12} \\ \mathtt{n21} & \mathtt{n22} \end{pmatrix} \begin{pmatrix} \mathtt{u11} & \mathtt{u12} \\ \mathtt{u21} & \mathtt{u22} \end{pmatrix}.$$

The `a`, `b` and `c` may be ignored here.

B.3 Calculations for U_3 with $\kappa(x) = x^3$.

Here we give all the calculations for U_3 with $\kappa(x) = x^3$.

We begin by initialising the memory allocation of the PARI process and defining functions `hij(x, y, k)` where `i`$\neq$`j` take the values $0, 1, 2$; these correspond to our rational functions for the $\varepsilon_{i,j}$ for the case of $\kappa(x) = x^k$.

```
allocatemem(1024000000)

h02(x, y, k) = (-(x3/4) - 1/4)^(k - 1)/(-(x3/4) - 1/4 - \
(x3/7 - 1/7)*x*y - (2*y)/7)

h01(x, y, k) = ((-(x3/4) + 1)*x - (3*x3)/4 - 1/2)^(k - 1)/ \
((-(x3/4) + 1)*x - (3*x3)/4 - 1/2 - (3*x3*x*y)/10 - (-(x3/10) + 1/5)*y) + \
((x3/4 - 1)*x + x3/4)^(k - 1)/((x3/4 - 1)*x + x3/4 + (x3/10 + 1/5)*x*y - \
(x3/10 + 1/5)*y)

h10(x, y, k) = (2*x3 + 2)^(k - 1)/(2*x3 + 2 - (-5*x3 + 5)*x*y + 4*y)

h12(x, y, k) = (5/8*(4 - x3)*x - 1/4*(3*x3 + 2))^(k - 1)/ \
(5/8*(4 - x3)*x - 1/4*(3*x3 + 2) - (15*x3*x*y)/14 + 1/7*(x3 - 2)*y) + \
(5/8*(x3 - 4)*x + x3/4)^(k - 1)/(5/8*(x3 - 4)*x + x3/4 + \
5/14*(x3 + 2)*x*y - 1/7*(x3 + 2)*y)

h21(x, y, k) = (2*x3 + 2)^(k - 1)/(2*x3 + 2 + 7/5*(x3 - 1)*x*y + (8*y)/5)

h20(x, y, k) = (7/2*(x3 - 4)*x + 6*x3 + 4)^(k - 1)/ \
(7/2*(x3 - 4)*x + 6*x3 + 4 + (21*x3*x*y)/2 + (4 - 2*x3)*y) + \
(7/2*(4 - x3)*x - 2*x3)^(k - 1)/(7/2*(4 - x3)*x - 2*x3 - \
7/2*(x3 + 2)*x*y + (2*x3 + 4)*y)
```

Next we compute the matrices corresponding to each rational function. We omit this output for brevity's sake.

```
m02 = matrix(6, 6, i, j, (polcoeff(polcoeff(h13(x, y, 3) + \
O(x^10), i-1, x) + O(y^10), j-1, y))%(x3^2 + 2));
m01 = matrix(6, 6, i, j, (polcoeff(polcoeff(h12(x, y, 3) + \
O(x^10), i-1, x) + O(y^10), j-1, y))%(x3^2 + 2));
m10 = matrix(6, 6, i, j, (polcoeff(polcoeff(h21(x, y, 3) + \
O(x^10), i-1, x) + O(y^10), j-1, y))%(x3^2 + 2));
m12 = matrix(6, 6, i, j, (polcoeff(polcoeff(h23(x, y, 3) + \
O(x^10), i-1, x) + O(y^10), j-1, y))%(x3^2 + 2));
m21 = matrix(6, 6, i, j, (polcoeff(polcoeff(h32(x, y, 3) + \
O(x^10), i-1, x) + O(y^10), j-1, y))%(x3^2 + 2));
m20 = matrix(6, 6, i, j, (polcoeff(polcoeff(h31(x, y, 3) + \
```

```
O(x^10), i-1, x) + O(y^10), j-1, y))%(x3^2 + 2));
```

Next, we form the matrix of U_3, which we call `A`

```
zero = matrix(6, 6, i, j, 0)
r1 = mattranspose(concat(concat(zero, m01), m02))
r2 = mattranspose(concat(concat(m10, zero), m12))
r3 = mattranspose(concat(concat(m20, m21), zero))

A = mattranspose(concat(r1, concat(r2, r3)))
```

We calculate the "characteristic power series" of this finite 18×18 matrix, `A`. Note that it really does make sense to consider such finite truncations of the matrices of such operators in view of Proposition 1.12, part 3. Next we set

```
x3 = sqrt(-2 + O(3^200))
```

— the 3-adic approximation to the $\sqrt{-2} \in \mathbb{Z}_3$, and then compute

```
l = matdet(A - t*matid(18));
```

and the slopes of this are calculated as follows:

```
newtonpoly(l, 3)
```

The output of this is

```
[2147483647, 2147483647, 12, 8, 11/2, 11/2, 5, 9/2, 9/2, 7/2, 7/2, 3, \
5/2, 5/2, 3/2, 3/2, 1/2, 1/2]
```

The output is presented in reverse to our definition in Chapter 1.

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