Notes on inner twists.

Kevin Buzzard

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There's a weight 2 level 243 trivial character cuspidal normalised eigenform f whose q-expansion looks like

$$q + aq^2 + 4q^4 - aq^5 + 2q^7 + 2aq^8 - 6q^{10} + aq^{11} - q^{13} + 2aq^{14} + 4q^{16} - \cdots$$

with a a square root of 6. It looks from the q-expansion that every coefficient is either an integer, or an integer multiple of a, and this is indeed the case. Moreover, the integer multiples look like they're happening for q^n with $n = 1$ mod 3, and the multiples of a look like they're happening for $n = 2$ mod 3, and this is also correct. We can see this by considering the twist of the form by the Dirichlet character χ of conductor 3—we get another eigenform $f \otimes \chi$, of level at most $243 * 9$, and by checking all possibilities we see that the only eigenform whose q-expansion agrees with the twist of $f \otimes \chi$ for the first few terms is f^σ, the Galois conjugate of f, where $1 \neq \sigma \in \mathrm{Gal}(\mathbf{Q}(\sqrt{6}/\mathbf{Q}))$.

This is an example of a form with an "inner twist". But this is a rather "generic" form too. For there is no CM involved (it is not the case that 50 percent of the a_p vanish—indeed $p = 3$ and $p = 89$ are the only primes less than 100 for which a_p vanishes) and furthermore there is no $\mathbf{Q}$-curve involved either: the 2-dimensional abelian variety $A/\mathbf{Q}$ associated to f does not have the property that over $k := \mathbf{Q}(\sqrt{-3})$, the field corresponding to χ, A splits (up to isogeny) as the product of two elliptic curves. What is happening, I think, is that A has an interesting endomorphism ring.

Some general result of Shimura shows that $\mathrm{End}^0_{\mathbf{Q}}(A)$ will be $E := \mathbf{Q}(\sqrt{6})$, the coefficient field of the modular form, so we get 2-dimensional λ-adic Galois representations attached to f, for λ running through the primes of E. But over k there are more endomorphisms: indeed, Cremona shows that $\mathrm{End}^0_{\overline{\mathbf{Q}}}(A) = \mathrm{End}^0_k(A)$ is a quaternion algebra $(-3, 6/\mathbf{Q})$, and 6 is not a norm for $\mathbf{Q}(\sqrt{-3})$, because the conic $A^2 + 3B^2 - 6C^2$ has no $\mathbf{Q}$-points, so this quaternion algebra does not split and an easy calculation (check all possibilities) shows that A must hence be absolutely simple. I think that the quaternion algebra must be the one of discriminant 6: if I've understood correctly it will be split by $\mathbf{Q}(\sqrt{6})$ and hence must be indefinite, at any rate.

Now here's a funny thing. Let λ be a prime of E and let $p \neq 3$ be a prime not dividing the norm of λ. Let's consider the λ-adic representation attached to f. If $p = 1$ mod 3 then the char poly of Frob_p will be $x^2 - tX + p$ with t an integer, and if $p = 2$ mod 3 then it will be $x^2 - t\sqrt{6}x + p$ again with t an integer. If the eigenvalues of this latter matrix are α and β, then $\alpha + \beta = t\sqrt{6}$ and $\alpha\beta = p$, so $\alpha^2 + \beta^2 = 6t^2 - 2p \in \mathbf{Z}$, and we see that the Galois representation restricted to G_k, the absolute Galois group of k, has trace in Z_ℓ, where $\lambda|\ell$. Note that if ℓ splits in E then the full Galois represetation attached to the abelian variety is taking values in $\mathrm{GL}_2(\mathbf{Q}_\ell)$ and hence the restriction to G_k is too. But if ℓ is inert in E then the Galois representation is taking values in $\mathrm{GL}_2(\mathbf{Z}_{\ell^2})$ and it's not clear to me whether one can tease it into $\mathrm{GL}_2(\mathbf{Z}_\ell)$. Andrei Yafaev told me that the Mumford-Tate group of the abelian variety will be $D^\times$. This sounds very right but I can't prove it. Is the Mumford-Tate group the centralizer of $D^\times$ in GSp_4?

Here's my guess: the centralizer of $D^\times$ is just something isomorphic to $D^\times$. I think this because I'm pretty sure that $D \otimes D = M_4(\mathbf{Q})$ and this gives two commuting actions of $D^\times$ on a 4-dimensional vector space. I can't find a pairing preserved by this though.

So I am guessing that the Mumford-Tate group is $D^\times$, so my guess is that the Galois representation attached to the modular form over the im quad field has image commensurable with $(\mathcal{O}_D \otimes \mathbf{Z}_\ell)^\times$, meaning that there will only be problems at the primes 2 and 3.

1 Remarks on the mod p Galois representations.

I just noticed that if the Mumford-Tate group is $B^\times$ then this forces the mod p representation of the absolute Galois group of k to be reducible if B ramifies at p. This is because $\mathcal{O}_B$, when tensored up to the integers $\mathcal{O}$ in a quadratic extension of $\mathbf{Q}_p$, doesn't become the full maximal order in $M_2(\mathcal{O})$. Hence one expects the mod 2 and mod 3 representations attached to the form to be reducible when restricted to the imaginary quadratic field.

The mod 2 representation attached to f takes values in $\mathrm{GL}_2(\mathbf{Z}/2\mathbf{Z})$ and a bit of experimenting shows that the splitting field of the representation is the Galois closure of $\mathbf{Q}(6^{1/3})$ (modulo the bad primes 2 and 3, the coefficient of q^p should be 1 mod 2 iff p is 1 mod 3 and doesn't split completely in $\mathbf{Q}(6^{1/3})$, that is, iff 6 has no cube root mod p. This representation is irreducible but when restricted to k of course becomes reducible.

The mod 3 representation attached to the form over $\mathbf{Q}$ is just 1 plus cyclo, so is certainly reducible over k as well.