Hochschild-Serre Spectral Sequence

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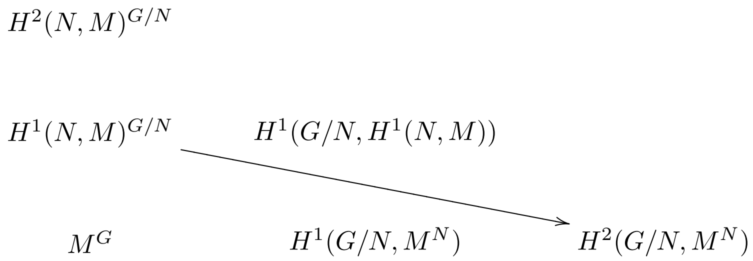
Let me try and push inf-res as far as it can reasonably go.

If N is a normal subgroup of a group G, and M is an abelian group with an action of G, then we have a spectral sequence

$$E_2^{i,j} = H^i(G/N, H^j(N,M)) \Rightarrow H^{i+j}(G,M).$$

Part of the associated big diagram is



and the associated long exact sequence of terms of low degree starts

$$0 \to H^1(G/N, M^N) \to H^1(G,M) \to H^1(N,M)^{G/N} \to H^2(G/N, M^N) \to$$

$$\to \ker\left(H^2(G,M) \to H^2(N,M)\right) \to H^1(G/N, H^1(N,M)) \to \ker\left(H^3(G/N, M^N) \to H^3(G,M)\right).$$

That's supposed to be one long exact sequence but it didn't fit on one line – there's a map $H^2(G/N, M^N) \to \ker\left(H^2(G,M) \to H^2(N,M)\right)$ and the whole sequence (seven non-zero terms) is supposed to be exact. Note that $\ker\left(H^2(G,M) \to H^2(N,M)\right)$ is the kernel of the obvious restriction map. The proof that this sequence is exact is just an easy exercise, using the Hochschild–Serre spectral sequence.

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Note that even if $G = N \times K$, a product, then *I have been told that* the spectral sequence may not degenerate!