Hodge theory for Buzzards.

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February 9, 2012

Written 2/10/07. Edited 19/01/10.

What are these notes?

They are notes aimed to explain certain very basic aspects of Hodge theory to someone who is completely happy with spectral sequences and hypercohomology and stuff, but who has no real understanding of real differential forms on a manifold. Someone like me, in fact. It started off as waffle (me collecting facts from various places) but has basically turned into a sketchy idea of a proof of how the comparison isomorphism between Betti and de Rham cohomology over the complexes behaves under various "complex conjugations" that exist when the underlying smooth projective variety is defined over the reals, and some related waffle.

1 Poincaré Lemmas.

If X is a smooth real manifold then we can consider the sheaves of C^∞ real-valued n-forms $\mathcal{A}^n$ on X. If X is a smooth complex manifold then we can consider the sheaf of complex-valued C^∞ (p,q)-forms $\mathcal{A}^{p,q}$ and the (complex-valued) holomorphic p-forms Ω^p. There appear to be at least three Poincaré Lemmas:

(1) (real Poincaré Lemma) On a smooth real manifold,

$$0 \to \mathbf{R} \to \mathcal{A}^0 \to A^1 \to A^2 \to \cdots$$

is exact, where $\mathbf{R} \to \mathcal{A}^0$ is the inclusion, and the maps $A^i \to A^{i+1}$ are all d.

(2) ($\overline{\partial}$ Poincaré Lemma, a.k.a. Dolbeaut Lemma) On a smooth complex manifold,

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \to \mathcal{A}^{p,1} \to \cdots$$

is exact, where $\Omega^p \to \mathcal{A}^{p,0}$ is the inclusion and the maps $\mathcal{A}^{p,i} \to \mathcal{A}^{p,i+1}$ are all $\overline{\partial}$.

(3) (Holomorphic Poincaré Lemma). On a smooth complex manifold,

$$0 \to \mathbf{C} \to \Omega^0 \to \Omega^1 \to \cdots$$

is exact, where $\mathbf{C} \to \Omega^0$ is the inclusion and $\Omega^i \to \Omega^{i+1}$ is d.

The proofs are mostly in Griffiths and Harris. (1) and (2) are stated on p38 and (3) is mentioned in passing on p448.

Now all the $\mathcal{A}^{p,q}$s and $\mathcal{A}^n$s are flabby, or perhaps "fine" in the sense that they admit partitions of unity in the sense of p42 of Griffiths-Harris. Hence all of their higher cohomology vanishes. So (applying the spectral sequence to compute cohomology of a complex, if you like, and noting that it degenerates at E_1 for trivial reasons) (1) tells us that nth Betti cohomology with coefficients in $\mathbf{R}$ is (global) closed n-forms modulo exact ones, which is of course the nth de Rham cohomology of X[1], (2) tells us that if X is a complex manifold then $H^q(X,\Omega^p)$ is "Dolbeaut Cohomology"

[1] I am sure that the induced isomorphism between Betti to de Rham cohomology will be the one that sends an i-form in de Rham cohomology to the linear map on i-chains given by integrating the i-form, although "this needs checking". Note that I've "explicitly" written down the map from de Rham to Betti; the general cohomological nonsense gave me a map the other way.

$H^{p,q}_{\overline{\partial}}(X)$ (defined to be the vector space with the property that (2) implies this fact). Is Ω^i flabby? I guess not: $H^1(\Omega^i)$ may well not vanish. So (3) only tells us that nth Betti cohomology with coefficients in $\mathbf{C}$ coincides with hypercohomology with coefficients in the complex of holomorphic differential forms, and hence that there is a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p) \implies H^{p+q}(X, \mathbf{C}).$$

Reminder: here X is a complex manifold and Ω^p is a sheaf on X with the analytic topology. Furthermore, the holomorphic Poincaré Lemma does not say anything about whether the spectral sequence degenerates—this is a serious part of the Hodge theorem. I guess the lemma does tell us that for a complex manifold X, we have a natural filtration on $H^n(X, \mathbf{C})$. But the Hodge Theorem is about compact Kähler manifolds (a Kähler manifold is a complex manifold with a nice metric on it) so hence applies to smooth projective varieties over the complexes (the Fubini-Study metric is a nice metric on projective space and it induces a nice metric on closed subspaces) but not to, say, affine varieties. Indeed if X is the complex numbers then $H^0(X, \Omega^1)$ (the holomorphic 1-forms on X with the analytic topology) is uncountably infinite-dimensional but $H^1(X, \mathbf{C}) = 0$.

2 Linear algebra.

First let me repeat Deligne's observation (1.2.6) from his "Hodge II" paper (IHES 1971). It works for an arbitrary abelian category but let's not go there, let's just consider a finite-dimensional vector space V over an arbitrary field k. For simplicity let's say that a *filtration* is just a decreasing filtration on V (that is, a filtration indexed by $\mathbf{Z}$ such that the subspaces V^m get smaller as m gets bigger) with the property that $V^N = 0$ for N sufficiently large and $V^N = V$ for N sufficiently small. Deligne calls these "filtrations finies". Deligne remarks that any subquotient of V gets a natural induced filtration; for subobjects one just intersects, for quotients one pushes forwards, and for subquotients one checks that the two constructions (writing the subquotient as a quotient of a sub, or as a sub of a quotient) coincide.

If V has a filtration $F = (F^m(V))$ then the associated graded vector space is $\mathrm{Gr}_F(V)$, the direct sum over m of $\mathrm{Gr}_F^m(V)$, where $\mathrm{Gr}_F^m(V) = F^m(V)/F^{m+1}(V)$.

Now say V has *two* filtrations F and G. Then $\mathrm{Gr}_F^m(V)$ inherits a G-filtration and one can speak of $\mathrm{Gr}_G^n(\mathrm{Gr}_F^m(V))$. Deligne shows (Zassenhaus butterfly) that there's a natural isomorphism $\mathrm{Gr}_G^n(\mathrm{Gr}_F^m(V)) = \mathrm{Gr}_F^m(\mathrm{Gr}_G^n(V))$.

Definition. Say F and G are two filtrations on V, and n is an integer. We say that F and G are *n-opposed* if $\mathrm{Gr}_G^a(\mathrm{Gr}_F^b(V)) = 0$ for $a + b \neq n$.

Now say V is a bigraded vector space, that is $V = \oplus_{a,b} V^{a,b}$ where we make the assumptions that V is finite-dimensional, and $V^{a,b} \neq 0$ for only finitely many pairs (a, b). Say furthermore that V has *weight* n, that is that $V^{a,b} = 0$ if $a + b \neq n$. We can now construct two n-opposed filtrations on V by setting $F^p(V) = \sum_{p' \geq p} V^{p',q}$ and $G^q(V) = \sum_{q' \geq q} V^{p,q'}$. Conversely, given two n-opposed filtrations F and G on V, one can construct a bigrading of weight n on V by setting $V^{a,b} = 0$ for $a + b \neq n$ and $V^{a,b} = F^a(V) \cap G^b(V)$ for $a + b = n$.

Proposition. (Deligne Hodge II, 1.2.6) The above constructions give an equivalence of categories between the category of bigraded vector spaces of weight n and vector spaces equipped with two n-opposed filtrations.

3 What does the Hodge theorem say?

Let V be a smooth projective algebraic variety over the complexes. Then GAGA tells us that sheaf cohomology of the Kähler differentials on the variety with its Zariski topology will coincide with sheaf cohomology of the holomorphic differentials with coefficients in the usual topology. That and the holomorphic Poincare lemma tells us that we have a (Hodge–De-Rham) spectral sequence

$$E_1^{p,q} = H^q(V, \Omega^p_{V/\mathbf{C}}) \implies H^{p+q}(V(\mathbf{C}), \mathbf{C})$$

where the right hand side is Betti cohomology. Note in particular that the right hand side is naturally $H_B^{p+q}(V(\mathbf{C}),\mathbf{R})\otimes_{\mathbf{R}}\mathbf{C}$ or $H_{dR}^{p+q}(V(\mathbf{C}),\mathbf{R})\otimes_{\mathbf{R}}\mathbf{C}$ (singular or classical (rather than algebraic) de Rham cohomology over the reals; they are canonically isomorphic via the comparison isomorphism, which works fine over the reals) and hence the right hand side has a semilinear complex conjugation on it, which we denote by a bar.

Theorem.

(A) The spectral sequence degenerates at E_1, so the natural filtration on $H^n(V(\mathbf{C}),\mathbf{C})$ has subquotients isomorphic to $H^i(V,\Omega^j)$ with $i+j=n$ [the sub is $H^0(V,\Omega^n)$ by the way].

(B) If $\overline{F}$ is the complex conjugate filtration, then F and $\overline{F}$ are n-opposed.

There was some debate over lunch today as to whether this *was* the Hodge theorem or whether it was just implied by it.

4 Varieties over the reals and algebraic de Rham cohomology.

If M is a real manifold then one can do de Rham cohomology with real-valued differential forms and get some cohomology groups and there's a comparison isomorphism (integration) relating these groups to Betti cohomology groups with coefficients in the reals. This is simply the classical Poincaré Lemma, which says that on a smooth real manifold the locally constant sheaf $\mathbf{R}$ is quasi-isomorphic to the complex of real-valued C^∞ differential forms with d as the maps.

If V is a smooth projective algebraic variety over the reals then one can form hypercohomology of the algebraic n-forms and get something called the *algebraic de Rham cohomology of* V. One can also compute the de Rham cohomology of $V(\mathbf{C})$ with real-valued forms. The resulting vector spaces are real vector spaces of the same dimension, both canonically isomorphic to $H^*(V(\mathbf{C}),\mathbf{C})$ when you tensor up to $\mathbf{C}$, but **they are not themselves canonically isomorphic.** For example, the analytic definition only depends on the manifold $V(\mathbf{C})$. Before I try and unravel exactly how they both fit into the singular cohomology with complex coefficients, let me explain two examples where one can see that the two de Rham cohomologies really do differ.

Firstly let V be $\mathrm{Spec}(\mathbf{R}[x]/(x^2+1))$, a perfectly good smooth projective variety over $\mathbf{R}$ (unless varieties are forced to be geometrically connected, in which case it's a perfectly good smooth scheme of finite type over $\mathbf{R}$). Now $V(\mathbf{C})=\{I,\overline{I}\}$, where I is the map from $A:=\mathbf{R}[x]/(x^2+1)$ to $\mathbf{C}$ sending x to i, and $\overline{I}$ is the other one. Now diff geom de Rham cohomology $H^0_{dR}(V(\mathbf{C});\mathbf{R})$, is just Betti cohomology with real coefficients so it's the real-valued functions on $V(\mathbf{C})$ (and hence 2-dimensional over $\mathbf{R}$). However algebraic de Rham cohomology H^0 is just $H^0(\mathcal{O}_V)=A$ by the spectral sequence, so also 2-dimensional over $\mathbf{R}$, and A is naturally the functions $f:V(\mathbf{C})\to\mathbf{C}$ such $f(\overline{I})=\overline{f(I)}$. Hence we get two different real subspaces in the (4-dimensional over $\mathbf{R}$) space of functions $V(\mathbf{C})\to\mathbf{C}$ and both, when tensored back up to $\mathbf{C}$, give the complex-valued Betti cohomology.

Secondly consider an elliptic curve $E/\mathbf{R}$ (note that E is geometrically connected) and let's do H^1. If $E(\mathbf{C})=\mathbf{C}/L$ with L a lattice then $H^1_B(E(\mathbf{C}),\mathbf{C})$ is $\mathrm{Hom}_{\mathbf{Z}}(L,\mathbf{C})=\mathrm{Hom}_{\mathbf{R}}(L\otimes_{\mathbf{Z}}\mathbf{R},\mathbf{C})=\mathrm{Hom}_{\mathbf{R}}(\mathbf{C},\mathbf{C})$ and diff geom H^1 over the reals is Betti over the reals so an analogous calculation shows that it's $\mathrm{Hom}_{\mathbf{R}}(\mathbf{C},\mathbf{R})$. Next note that $H^1_B(E(\mathbf{C}),\mathbf{C})$ is hypercohomology of the Kähler differentials over the complexes, so contains $H^0(E_{\mathbf{C}},\Omega^1_{E/\mathbf{C}})=\mathbf{C}.dz$, and the dictionary is that $t.dz$ corresponds to the map $\mathbf{C}\to\mathbf{C}$ given by multiplication by t. Now we have nailed $H^1_B(E(\mathbf{C}),\mathbf{C})=\mathrm{Hom}_{\mathbf{R}}(\mathbf{C},\mathbf{C})$, diff geom $H^1_{dR}=Hom_{\mathbf{R}}(\mathbf{C},\mathbf{R})$ and $H^0(E_{\mathbf{C}},\Omega^1_{E/\mathbf{C}})=Hom_{\mathbf{C}}(\mathbf{C},\mathbf{C})$, but not alg geom H^1_{dR}. However we do have enough to see that alg geom H^1_{dR} isn't diff geom H^1_{dR} for the following reason: diff geom H^1_{dR} has trivial intersection with $H^0(E_{\mathbf{C}},\Omega^1_{E/\mathbf{C}})$ because multiplication by t as a map $\mathbf{C}\to\mathbf{C}$ never has image in $\mathbf{R}$ unless $t=0$, whereas alg geom H^1_{dR} contains the line $H^0(E,\Omega^1_{E/\mathbf{R}})$ which also lies in $H^0(E_{\mathbf{C}},Omega^1_{E/\mathbf{C}})$. Conclusion: the two de Rham cohomologies can't coincide, because one is $\mathrm{Hom}_{\mathbf{R}}(\mathbf{C},\mathbf{R})$ and the other contains some non-zero element of $\mathrm{Hom}_{\mathbf{C}}(\mathbf{C},\mathbf{C})$.

Let me try and unravel this by putting ourselves back in the general situation ($V/\mathbf{R}$ smooth projective) and trying to reconstruct both de Rham cohomologies from $H^n(V(\mathbf{C}),\mathbf{C})$. The diff geom one is easy: it's just $H^n_B(V(\mathbf{C}),\mathbf{R})$. I guess that reconstructing the algebraic de Rham cohomology is called "descent".

5 Descent.

In the algebraic world here's the kind of question we want to answer. Given an algebraic variety V over a field K and a coherent sheaf M on V, and a separable quadratic extension L/K, one can base change V to L getting V_L, pull M back to V_L and (because L/K is flat, if you like,) we know $H^i(V_L,M_L)=H^i(V,M)\otimes_K L$.

Now how do we recover $H^i(V,M)$ from $H^i(V_L,M_L)$? Well if $f:X\to Y$ is a continuous map of locally ringed spaces and M is an $\mathcal{O}_Y$-module then there is a canonical map $H^i(Y,M)\to H^i(X,f^*M)$. Hence if $1\neq c\in\mathrm{Gal}(L/K)$ then there's a map $c:V_L\to V_L$ and a canonical isomorphism $c^*M_L\to M_L$ and so we get a map $H^i(V_L,M_L)\to H^i(V_L,c^*M_L)=H^i(V_L,M_L)$ and if we regard these cohomology groups as vector spaces over L then the first map is L-antilinear and the second (the equality) is L-linear, and we see that $H^i(V,M)$ is just the fixed points (we do if $i=0$ at least, and probably the general case follows). I'm sure the same arguments will work for hypercohomology too. It follows that we can reconstruct algebraic de Rham cohomology over the reals, from algebraic de Rham cohomology over the complexes, by unravalling this. We have $V/\mathbf{R}$ smooth projective. We consider the map $V_{\mathbf{C}}\to V_{\mathbf{C}}$ given by complex conjugation, and identify the pullback of the Kähler differentials with the Kähler differentials. We have to explicitly unravel what's going on and it suffices to do this on sections over some open; if U is an affine open in $V_{\mathbf{C}}$ then sections of $\Omega^1_{V_{\mathbf{C}}/\mathbf{C}}$ over $U=\mathrm{Spec}(A)$ are Kähler differentials $\Omega_{A/\mathbf{C}}$ and the idea is that given $d:A\to\Omega_{A/\mathbf{C}}$ we set $B=A\otimes_{\mathbf{C},c}\mathbf{C}$, the twist of A, and observe that as a set $\Omega_{B/\mathbf{C}}$ can be thought of as $\Omega_{A/\mathbf{C}}\otimes_{\mathbf{C},c}\mathbf{C}$. I think that we conclude that locally what's going on is that the algebraic de Rham cohomology is the stuff invariant under the involution sending the differential fdg to $\overline{(P\mapsto f(\overline{P})d(P\mapsto g(\overline{P})))}$. Now this makes sense in the holomorphic category, however because these maps are "moving the base" I don't think it makes sense to "take invariants" on the level of sheaves. However one notes that one can extend this funny involution to an involution on the constant sheaf $\mathbf{C}$ in the holomorphic category and $d:\mathbf{C}\to\mathcal{O}$ commutes with this action, as in $\mathcal{O}$ we're considering the endomorphism sending $P\mapsto f(P)$ to $P\mapsto\overline{f(\overline{P})}$ so on $\mathbf{C}$ it's just complex conjugation and moving the base, and lo and behold we have a persuasive argument that on Betti cohomology we can recover algebraic de Rham cohomology as the invariants of the endomorphism of $H^n(V(\mathbf{C}),\mathbf{C})$ under the semilinear map obtained by doing complex conjugation on $V(\mathbf{C})$ *and* on $\mathbf{C}$. Tony Scholl would call F_∞ the map $V(\mathbf{C})\to V(\mathbf{C})$ and would refer to the semilinear map whose invariants are algebraic de Rham cohomology as $\overline{F}_\infty$. I think that this must be a *very sketchy* proof of the fact that under the comparison isomorphism

$$H^n_B(V(\mathbf{C}),\mathbf{C})=H^n_B(V(\mathbf{C}),\mathbf{R})\otimes_{\mathbf{R}}\mathbf{C}\to H^n_{dR}(V_{\mathbf{C}})=H^n_{dR}(V/\mathbf{R})\otimes_{\mathbf{R}}\mathbf{C},$$

the semilinear maps $F_\infty\otimes c$ on the left and $1\otimes c$ on the right coincide.

6 Links with representations of Weil groups.

If a rank n motive over $\mathbf{Q}$ is supposed to be the same thing as an isobaric automorphic representation of GL_n, and one wonders what the local dictionary is, one might make the following guesses. At a finite place p, choose some prime $\ell\neq p$ and consider the ℓ-adic representation attached to the motive. Restricting to the local Galois group gives an n-dimensional ℓ-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and hence, by Grothendieck's construction, a Weil-Deligne representation. One hopes it's semi-simple; one then gets a π_p via local Langlands.

But what about the infinite place? We need a π_∞, and by local Langlands at infinity we need an n-dimensional representation of the Weil group of the reals. Betti cohomology is however only providing us with a representation of the absolute Galois group of the reals.

Well, here's the dictionary. Say we have $V/\mathbf{R}$ smooth projective. Its Betti cohomology with complex coefficients $H := H^n(V(\mathbf{C});\mathbf{C})$ has a $\mathbf{C}$-linear endomorphism F_∞ induced by $F_\infty : V(\mathbf{C}) \to V(\mathbf{C})$. It also has a bigrading $H = \oplus_{p,q} H^{pq}$ and F_∞ is a linear map which sends H^{pq} to H^{qp} (because $F_\infty \otimes c$ preserves the grading and $1 \otimes c$ switches it!). So now we simply define the action of $W_{\mathbf{R}}$ on H thus: $u \in \mathbf{C}^\times$ acts on H^{pq} as multiplication by $u^{-p}\overline{u}^{-q}$, and j (the thing of order 4 whose square is -1) acts on H^{pq} as the map $i^{p+q}(F_\infty \otimes 1)$. Note: we're clearly not getting all iso classes of Weil group representations here! Presumably we're getting the ones that Clozel calls "algebraic"?