What the Harish-Chandra homomorphism looks like.

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February 9, 2012

Last modified 02/09/2010.

1 Introduction/summary.

Written 17/11/07. Tinkered with a little (fixed typos) in 2010. I think my source was Knapp's book on rep theory of semisimple groups.

The Harish-Chandra homomorphism is a way of completely identifying what the centre of the universal enveloping algebra of a complex reductive Lie algebra is. The centre is in fact always isomorphic to a polynomial ring in d variables, where d is the dimension of a Cartan subalgebra. The definition of the homomorphism depends on the choice of a notion of positivity but it has been cleverly normalised so that the homomorphism itself does not depend on this choice!

2 The homomorphism.

If $\mathfrak{g}$ is a complex reductive Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra (i.e. a "maximal torus" on the Lie algebra level) then $\mathfrak{h}$ is abelian so its universal enveloping algebra $\mathcal{H} := U(\mathfrak{h})$ is just a polynomial algebra. Harish-Chandra observed that if we choose an ordering and hence get positive roots $E_1, E_2, \ldots$, and let I denote the ideal $U(\mathfrak{g})E_1 + U(\mathfrak{g})E_2 + \ldots$, then $\mathcal{H} + I$ (within $U(\mathfrak{g})$) is a direct sum, the centre $Z(U(\mathfrak{g}))$ of $U(\mathfrak{g})$ is contained within $\mathcal{H} + I$, and that the projection $Z(U(\mathfrak{g})) \to \mathcal{H}$ along I was an injection. But much much better: if you then compose this projection with the algebra automorphism of $\mathcal{H}$ induced by sending $h \in \mathfrak{h}$ to $h - \delta(h)1 \in \mathcal{H}$ (with δ half the sum of the positive roots) then the resulting map $Z(U(\mathfrak{g})) \to \mathcal{H}$ was an injective ring homomorphism, with image precisely $\mathcal{H}^W$, the things fixed by the Weyl group. It's a general fact that $\mathcal{H}^W$ is isomorphic to a polynomial ring in $\dim(\mathfrak{h})$ variables.

Note that if $\mathfrak{z}$ is the centre of $\mathfrak{g}$ then $\mathfrak{z} \subseteq \mathfrak{h}$ and the induced map $U(\mathfrak{z}) \to Z(U(\mathfrak{g})) \to \mathcal{H}$ is the obvious one; the projection $Z(U(\mathfrak{g})) \to \mathcal{H}$ induces the identity on $U(\mathfrak{z})$, and the twist by δ doesn't change anything because δ doesn't move $\mathfrak{z}$. Standard algebra arguments show that the algebra maps $\mathcal{H}^W \to \mathbf{C}$ all come via restriction from algebra maps $\mathcal{H} \to \mathbf{C}$, and the maximal ideals of $\mathcal{H}^W$ are just the W-orbits of the maximal ideals of $\mathcal{H}$.

3 An example.

Let $\mathfrak{g}$ be $\mathfrak{gl}_2(\mathbf{C})$ with basis $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let the Cartan subalgebra be the span of H and Z. Now the root spaces are spanned by E and F, the roots are the characters of $\mathfrak{h}$ sending H to ± 2 and Z to zero, with E corresponding to the number $+2$. The un-normalised H-C homomorphism sends Z to Z. The adjoint representation of $\mathfrak{gl}_2$ (wrt the basis E, F, H, Z and with the algebra acting on the left) is this:

$$E \mapsto \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Z \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the associated Killing form (the one whose (i,j)th entry is the trace of $\rho(e_i)\rho(e_j)$ is

$$\begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which isn't non-degenerate, so I don't think reductive Lie algebras have Casimir elements! Had we worked with $\mathfrak{sl}_2$ one checks that we would have just thrown away the last row and column for both the adjoint representation and the Killing form, which is now non-degenerate with inverse

$$\begin{pmatrix} 0 & 1/4 & 0 \\ 1/4 & 0 & 0 \\ 0 & 0 & 1/8 \end{pmatrix}$$

and so if $X_1 = E$, $X_2 = F$ and $X_3 = H$ then the dual basis is $X^1 = F/4$, $X^2 = E/4$ and $X^3 = H/8$ and the Casimir element is $\sum_{i,j} g_{i,j} X^i X^j$ (with $g_{i,j}$ the Killing form) is $4X^1X^2 + 4X^2X^1 + 8(X^3)^2 = EF/4+FE/4+H^2/8$. Well, that's what I made it! Another way of writing it is $\sum X_i X^i$ and again I get $EF/4+FE/4+H^2/8$. Why is everyone else out by factors of 4 or 8 or whatever? Let's multiply by 8 just to clear the denominators, and set $C = 2EF + 2FE + H^2$ even though as far as I can see C is in fact eight times the Casimir element. The un-normalised Harish-Chandra homomorphism is easily evaluated on C: one has $2EF = 2H + 2FE$ so $C = H^2 + 2H + 4FE$ and by definition this gets sent to $H^2 + 2H$. Now to normalise it we have to compose with the algebra automorphism of the polynomial ring $\mathbf{C}[Z,H]$ sending Z to Z and H to $H - \delta(H) = H - 1$ (the positive root is E and the associated linear map on the Cartan sends H to 2 and Z to zero). So we get $(H-1)^2 + 2(H-1) = H^2 - 1$. Hence the normalised Harish-Chandra homomorphism sends Z to Z and C to $H^2 - 1$, and the theorem is that the map is injective with image the subring of $\mathbf{C}[Z,H]$ fixed by the Weyl group, and the Weyl group sends H to $-H$ and fixes Z so the image is $\mathbf{C}[Z,H^2]$ and lo and behold we've proved that the centre of the universal enveloping algebra is generated freely as a polynomial ring by Z and C.