Grossencharacters.

Kevin Buzzard

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Let K be a number field; no doubt everything I say here is either trivially true or can trivially be modified in the function field setting but I don't care about that today. Let $\mathcal{O}_K$ denote the integers of K and $\mathbf{A}_K$ the adeles of K.

Definition. *A* Grössencharacter *is a continuous group homomorphism*

$$K^\times \backslash \mathbf{A}_K^\times \to \mathbf{C}^\times.$$

Equivalently (by definition of the quotient topology), a grössencharacter is a continuous group homomorphism $\mathbf{A}_K^\times \to \mathbf{C}^\times$ with $K^\times$ in its kernel.

Example 1) Let's recall how one normalises norms. If K is a finite extension of the p-adics or reals then K has an additive Haar measure dx and we define $|a|$ by $|a|dx = d(ax)$. This is a completely canonical norm on such a field. Explicitly: if K is a finite extension of the p-adics then define a norm on K by letting the norm of a uniformiser be the reciprocal of the size of the residue field. If K is the reals, then put the usual norm on it. If K is the complexes, put the square of the usual norm on it (the "usual" norm being the one where $|2| = 2$). Define the global norm $||x||$ of an idele x as the product of the local norms. Then the norm of an element of $K^\times$ is 1 and so $||.||$ defines a Grössencharacter (which takes values in the positive reals).

Example 2) If t is a positive real and z is a complex number, then t^z is of course defined to be $\exp(z\log(t))$. So $||.||^z$ is also a Grössencharacter.

Example 3) If χ is a Grössencharacter, then ψ defined by $\psi(x) = |\chi(x)|$ is one too, where here $|.|$ denotes the usual norm on the complexes. For example if χ is $||.||^z$ then $|\chi|$ is $||.||^{\mathrm{Re}(z)}$.

Lemma. *If χ is a Grössencharacter and the image of χ is contained in the positive reals, then $\chi = ||.||^\sigma$ for some real σ.*

Proof. It suffices to check this for a subgroup of $K^\times \backslash \mathbf{A}_K^\times$ of finite index, because then the quotient will be a finite order character taking values in the positive reals, which has no non-trivial elements of finite order. So we check it on $U_K \backslash \left(\widehat{\mathcal{O}}_K^\times K_\infty^\times\right)$ where U_K denotes the units of K. This is of course a subgroup of finite index, the index being at most the class group. Any continuous character from the units of a local field to $\mathbf{C}^\times$ must be trivial on a subgroup of finite index, and hence if it takes only positive real values, must be trivial. So we just have to check that the only continuous characters $K_\infty^\times \to \mathbf{R}_{>0}$ trivial on the units are powers of the norm. Well, the only continuous characters $\mathbf{C}^\times \to \mathbf{R}_{>0}$ are trivial on all roots of unity and hence on the unit circle, so factor through the norm, and now we're done by the units theorem and the fact that the using exp and log, the only continuous maps $\mathbf{R}_{>0} \to \mathbf{R}_{>0}$ are of the form $x \mapsto x^\sigma$ for σ real. $\square$

Definition. *A Grössencharacter is* unitary *if it takes values in the unit circle.*

Corollary. *If χ is any Grössencharacter then there's a unique real number σ such that $\chi/(||.||^\sigma)$ is unitary.*

A good "concrete" reference for unitary Grossencharacters is Miyake's book on modular forms, although it's not written in the adelic language so one has to use Tate's dictionary in Cassels-Froehlich to pass to the adelic language; Miyake deals with characters on groups of fractional ideals and calls them Hecke characters; Tate's dictionary gives a unitary Grössencharacter for each of Miyake's Hecke characters.

At a real infinite place the only possible choices for a unitary grössencharacter are $x \mapsto \mathrm{sgn}(x)^u|x|^{iv}$ for $u \in \{0,1\}$ and $v \in \mathbf{R}$; at a complex place we must be of the form have $x \mapsto (x/|x|)^u|x|^{iv}$ for $u \in \mathbf{Z}$ and $v \in \mathbf{R}$. Furthermore, for the units to be killed by such a map, one checks easily that the sum of the vs must be 0. Miyake writes down the functional equation for the L-function for such a Grössencharacter; the Gamma factors at the real infinite places are $\Gamma((s+iv+|u|)/2)$ and at the complex infinite places are $\Gamma(s+iv+(|u|/2))$. The functional equation relates $\Lambda(\psi,s)$ to $\Lambda(\overline{\psi},1-s)$.

Finally, let χ be $\chi_0.||.||^\sigma$ with χ_0 unitary (this is the most general kind of Grössencharacter, by the Corollary). The $||.||^\sigma$ just shifts everything along by σ, as it were; one finds that $L(\chi,s) = L(\chi_0, s-\sigma)$, the Gamma factors look like $\Gamma((s-\sigma+iv+|u|)/2)$ and $\Gamma(s-\sigma+iv+(|u|/2))$, and the functional equation relates $\Lambda(\chi,s)$ to $\Lambda(\overline{\chi},2\sigma+1-s)$.

I should write down the L-function more precisely but am too lazy. See Miyake. There are other fudge factors coming from 2s and 2πs at the infinite places, and local L-factors at the finite places; also a Gauss sum in the functional equation.