On the classical Satake isomorphism.

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Introduction

These notes are an attempt to fathom out exactly how "functorial" the (classical normalisation of the) "Satake correspondence" (that is, the unramified local Langlands correspondence) is, and how much it depends on Frobenius conventions.

1 Ingredients for the Satake correspondence.

There are really two ingredients: one gives an isomorphism between a non-archimedian Hecke algebra coming from a hyperspecial maximal compact and the Weil group invariants of another Hecke algebra associated to a torus, and the other interprets Weyl group orbits on the aforementioned torus as certain twisted semisimple conjugacy classes. I will start by writing down definitions and taking both of these things apart.

2 L-groups.

We basically follow Kottwitz in our definition of the L-group. If G is connected and reductive over a separably closed field, then its based root datum is a projective limit $(X^*, \Delta^*, X_*, \Delta_*)$ of the character/cocharacter groups etc associated to a choice of a Borel and torus; the idea is that X^* isn't canonically the character group of any particular torus, but if one chooses a torus *and a Borel* in G then X^* becomes canonically isomorphic to the character group of the torus. The dual group of G is a pair consisting of a group $\mathrm{G}^\vee$ (over $\mathbf{C}$, say) plus an isomorphism between the based root datum of $\mathrm{G}^\vee$ and the dual of the based root datum of G. Note that $\mathrm{G}^\vee$ is "only defined up to inner automorphism" in some sense: given two dual groups ${\mathrm{G}^\vee}_1$ and ${\mathrm{G}^\vee}_2$ for $\mathbf{G}$, they have canonically isomorphic based root data, but there is no canonical map ${\mathrm{G}^\vee}_1 \to {\mathrm{G}^\vee}_2$; isomorphisms exist, and furthermore isomorphisms which induce the canonical identification between the based root data exist, however more than one such thing exists in general, although any two such will differ by an inner automorphism.

Now say G is connected reductive over an arbitrary field k and let Γ denote the absolute Galois group of k (note that Γ is only defined up to inner automorphisms usually, but we fix a separable closure of k so Γ really is well-defined). General nonsense gives us an action of Γ on the based root datum attached to G and hence an action of Γ on the based root datum attached to $\mathrm{G}^\vee$. If we choose a pinning of $\mathrm{G}^\vee$ then this lifts the action of Γ to $\mathrm{G}^\vee$ and we may form the semidirect product: this is the L-group of G. My guess is that the formal definition should be this: the L-group sits in the middle of a short exact sequence with the dual group as the sub and the Galois group as the quotient, and we know that the sequence splits and furthermore we're given a preferred $\mathrm{G}^\vee(\mathbf{C})$-conjugacy class of splitting. Let's go with this for the moment; I'm not 100 percent sure it's right though.

3 Cartier Corvallis section 3 a la Cartier.

Let me stick with Cartier's notation. I will redo this section with more sensible notation in a minute.

Assume G is connected reductive over a non-arch local field F, and furthermore assume G is unramified. Note that Cartier does not assume this from the outset, and hence Cartier's M is not a torus for a long time. The payoff is that he gets a Satake isomorphism that works if the max compact is special, not just hyperspecial. On the other hand, I don't care about this case. OK, so G unramified. Now let's enjoy the rollercoaster of Cartier's non-standard notation, and let's try and add some clarifying remarks.

Fact 1) All maximal F-split tori in G are conjugate.

So let A be a maximal F-split torus. Let M be its centralizer in G. So M is well-defined up to conjugation.

Fact 2) M is a torus.

This is because G is quasi-split. In general the derived subgroup of M is called the "anisotropic kernel" of M, and the theorem is that the anisotropic kernel of G is trivial iff G is quasi-split.

Fact 3) M is in fact a maximal torus in G.

This is obvious really. Just split G and use the fact that any torus is contained in a maximal torus.

Fact 4) There exists a Borel subgroup P, defined over F, and containing M.

This is because G is unramified.

Fact 5) We could have "done things the other way": we could have let P be a Borel over F, let M be a maximal torus in P, and then let A be the maximal split subtorus of M.

OK. We get a Weyl group associated to this situation: the normaliser of A, modulo M. If I've understood Borel correctly, this can be thought of as the subgroup of the Weyl group of M (over an alg closure of F) consisting of the things that stabilise A.

Before I press on let me note that for T a split torus over F, we have $X_*(T) = T(F)/T(\mathcal{O})$, and the completely canonical dictionary is this: think of $X_*(T)$ as the $\mathbf{Z}$-dual of $X^*(T)$ and now given an element ϕ of $X^*(T)$ and an element t of $T(F)$ I need to give you an integer: it's just the valuation of $\phi(t)$ (normalised, of course, so that the valuation of a uniformiser is 1).

Now let's break with usual notation and follow Cartier by letting $X^*(M)$ denote the maps $M \to \mathbf{G}_m$ which are *defined over* F; this just corresponds to the (saturated) subgroup of $X^*(M_{F^{\mathrm{sep}}})$ consisting of elements which are fixed by Γ_F. In fact let's let Q denote the maximal split torus quotient of M, so $X^*(M) = X^*(Q)$. Recall that A is the maximal split subtorus of M, so we have a map $A \to Q$ which will be an isogeny but not in general an isomorphism (for example if $G = U(1,1)$ then M is a non-arch "Deligne's S" and A and Q are both $\mathbf{G}_m$ but the map $A \to Q$ is squaring. We can hence identify $X^*(Q)$ with a finite index subgroup of $X^*(A)$.

Let $X_*(M)$ denote the $\mathbf{Z}$-dual of $X^*(M)$, so it's $X_*(Q)$. Because of saturatedess, the inclusion $X^*(M) \subseteq X^*(M_{F^{\mathrm{sep}}})$ corresponds to a surjection $X_*(M_{F^{\mathrm{sep}}}) \to X_*(M) = X_*(Q)$. Note that Galois acts on the character group $X_*(M_{F^{\mathrm{sep}}})$, and $X_*(A)$ is the Galois invariants.

The corresponding map $X_*(A) \to X_*(Q)$ identifies $X_*(A)$ with a finite index subgroup of $X_*(Q)$. Now there's a natural map $M(F) \to Q(F) \to X_*(Q) = X_*(M)$, the map $Q(F) \to X_*(Q)$ in the middle (Cartier calls it "ord") being the one factoring through $Q(F)/Q(\mathcal{O}) = X_*(Q)$ (because Q is split). Note that although $M \to Q$ is surjective, $M(F) \to Q(F)$ might not be. Let $\Lambda(M)$ denote the image of $M(F)$ in $X_*(M)$. One can check that $X_*(A)$ is a subgroup of $\Lambda(M)$—but then using a bit of Galois cohomology one can check that in fact $X_*(A) = \Lambda(M)$. Cartier doesn't seem to mention this (it might not be true in the generality in which he's working at the time—he hasn't assumed G unramified at this point in his exposition—but it's true if G is unramified: see 1.1 and 1.2 of some notes by Casselman at `http://www.netera.ca/seminars/math/macdonald.pdf`).

In particular what I'm saying here is that the injection $A \to M$ gives us an injection $A(F) \to M(F)$ and a bijection $X_*(A) = A(F)/^oA \to M(F)/^oM$, and that this bijection can be, if you like, be being thought of as taking place in $X_*(Q) = Q(F)/Q(\mathcal{O})$, where $X_*(A)$ is a finite index subgroup.

Let ^oM denote the kernel of $M(F) \to \Lambda(M)$ and let $^oA = A(\mathcal{O})$ denote the kernel of $A(F) \to X_*(A)$. An unramified character of M just means a map $M(F) \to \mathbf{C}^\times$ which factors through $\Lambda(M)$ (i.e., is zero on ^oM), so it's really a map $\Lambda(M) \to \mathbf{C}^\times$ and hence can be identified with a $\mathbf{C}$-point of the complex torus $A^\vee$. Cartier sets $T = A^\vee$ and $X = T(\mathbf{C})$.

Cartier defines $T' = M^\vee$, the dual complex torus. The inclusion $A \to M$ gives rise to a surjection $T' \to T$, although Cartier likes to think of points on T as unramified characters of M rather than of A, and note that given an unramified character of A it will take a little teasing to interpret it as an unramified character of M (because this relies on the Galois cohomology calculation). The point is that the natural map $A(F)/^oA \to M(F)/^oM$ is an isomorphism, but surjectivity needs checking (it's the vanishing of an H^1).

4 Cartier Corvallis section 3 summarised.

I'll just go over the previous section again.

G unramified over local F. Let A be a maximal F-split torus and let M its centraliser—a maximal torus of G, it turns out, because G is quasi-split. Let K be a hyperspecial maximal compact in $G(F)$. Let ^oM be $M(F) \cap K$ and let ^oA be $A(F) \cap K$. First I claim that the natural inclusion $A(F) \to M(F)$ induces an isomorphism

$$A(F)/^oA \to M(F)/^oM.$$

Note that if we were working instead with Q, the maximal split torus quotient of M, then the analogous statement wouldn't be true because $M(F) \to Q(F)$ isn't in general surjective (consider $M = \mathrm{Res}_{E/F}\,\mathrm{GL}_1$ for E the unramified quadratic extension).

Let W be the Weyl group for the pair (G, A), that is, $N(A)/Z(A)$ (the normaliser of A, modulo the centraliser). Note that $Z(A)$ is just M. Now W acts on A and it also acts on M, because anything that normalises A also normalises its centralizer (easy check, if I got this right). Casselman says that one can always choose representatives for W in K.

The Satake isomorphism is a map

$$H(G, K) \to H(M, {}^oM)$$

defined by sending $f \in H(G, K)$ to the function Sf, defined by

$$(Sf)(m) = \delta(m)^{1/2} \int_N f(mn)\mathrm{d}n = \delta(m)^{-1/2} \int_N f(nm)\mathrm{d}m.$$

This turns out to be an injective algebra homomorphism, with image equal to the W-invariants of $H(M, {}^oM)$. Note that, contrary to the way I always seem to remember it, S is *not* "restriction"—it's not that at all. Note also that the definition of δ is this: the group $M(F)$ acts on the Lie algebra of N (where $B = MN$) which is a vector space over F; take the determinant (hence a map $M \to F^\times$) and then the norm; this is δ.

Note finally that $H(M, {}^oM) = H(A, {}^oA)$ for trivial reasons. This latter is (by my notes on tori) the ring of functions on the algebraic variety $\widehat{A}$ over $\mathbf{C}$.

5 Cartier Corvallis section 4.

Section 4 of Cartier finally introduces a special maximal compact K of $G(F)$; then $^oM = M(F) \cap K$ is the kernel of the ord map, and $M(F)/^oM = \Lambda(M)$. Cartier (following Satake) shows that "averaging over N", multiplied by some appropriate half-the-sum-of-the-positive-roots thing, gives an isomorphism between $H(G, K)$ (the spherical Hecke algebra) and $H(M, {}^oM)^W = \mathbf{C}[\Lambda(M)]^W$ (an element of $\Lambda(M)$ corresponding to the locally constant function on $M(F)$ given by characteristic function of the corresponding subset of $M(F)$).

Note that Cartier doesn't give a monkeys about Frobenius conventions; this part of the story does the following: given an unramified π for G, this gives us a character of the spherical Hecke algebra and hence a W-orbit of group homomorphisms $\Lambda(M) \to \mathbf{C}^\times$, that is, a W-orbit of unramified characters of M, that is, of maps $M(F)/{}^oM \to \mathbf{C}^\times$. Because of this Galois cohomology calculation that isn't in Cartier, this can be thought of as a W-orbit of unramified characters of A, and hence an element of $A^\vee/W$.

To see that this part of the argument has nothing to do with Frobenius conventions, consider the case of G a split torus. Then Cartier is saying that unramified maps $G(F) \to \mathbf{C}^\times$ are elements of a dual torus, and this is right: an unramified map $G(F) \to \mathbf{C}^\times$ is just a map from $X_*(G)$ to $\mathbf{C}^\times$ and hence a map from $X^*(G^\vee)$ to $\mathbf{C}^\times$, but all such things are given by elements of $G^\vee$. There are no issues of convention here at all.

All of this needs rewriting. I seem to be saying the same things twice.

SUMMARY SO FAR: $H(G,K) = H(M, {}^oM)^W$, the map being "average over N and normalise by half the sum of the positive roots."

6 Borel Corvallis.

This is section 6 of Borel Corvallis. Let me stick with Cartier's definitions, so I'll have to first go through and give a dictionary. Borel's ${}_kW$ is Cartier's W, Borel's T is Cartier's M, Borel's T_d is Cartier's A, Borel's Y is my $A^\vee$. Borel has a map A (nothing to do with a torus) which depends on a choice of generator for the Galois group of the splitting field, but he only uses it to define tori U and V in his $T^\vee$ (Cartier's T'—but I'll refer to the torus in the L-group as $M^\vee$) and these tori are independent of choice of generator (so in particular it could be arithmetic or geometric Frobenius): indeed, $\ker(A)$ is the subgroup of $T^\vee$ fixed by Galois, and so $U = \ker(A)^o$ doesn't depend on σ, and $V = \ker(\nu)$ (I'll define ν in a second). Another way of seeing that V doesn't change when you change σ to σ^{-1} is to note that $\sigma(t)/t = \sigma^{-1}(s)/s$ for $s = \sigma(t^{-1})$. Borel's U is $Q^\vee$ for Cartier's Q, Borel's Y is Cartier's $A^\vee$, Borel's $\nu : T^\vee \to Y$ is induced by Cartier's inclusion $A \to M$ and the isogeny $U \to Y$ is just the dual of the isogeny $A \to Q$.

Having got over all that, Borel's bijections (6.4 and 6.5) identify W-orbits on Cartier's $A^\vee$ with twisted conjugacy classes—but the twisting can be by an arbitrary generator of the Galois group!! Borel just assumes that G is quasi-split over an arbitrary field k, and split over a finite Galois extension of k with cyclic Galois group. In particular, Borel seems to prove that for any generator σ of the Galois group of the splitting field, if we fix an element of $M^\vee$ (a maximal torus in the dual group) and consider the other elements of this torus in the same σ-conjugacy class as this, they coincide with the other elements of the torus in the same σ'-conjugacy class.

7 Summary so far.

G unramified over F local, A a maximal F-split torus, M its centraliser (a maximal torus in G), W the Weyl group for (G, A) (that is, the normaliser of A modulo the centralizer). Let K be a hyperspecial maximal compact, and let ${}^oM = M(F) \cap K$ and ${}^oA = A \cap K$ which I think is just $A(\mathcal{O})$. Then $H(G(F), K)$ is isomorphic to $H(M(F), {}^oM)^W = H(A(F), A(\mathcal{O}))^W$.

Furthermore, if σ is an arbitrary generator of $\mathrm{Gal}(E/F)$, with E a finite cyclic extension of F splitting G, then every σ-twisted semisimple conjugacy class in $\widehat{G}(\mathbf{C})$ hits $\widehat{M}$, and the image of such a thing in $\widehat{A}$ is well-defined in $\widehat{A}/W$. Note that this dictionary doesn't depend at all on the choice of σ!

8 The unramified local Langlands correspondence.

Our conclusion is that the correspondence depends on conventions. If we normalise our class field theory isomorphisms by sending a uniformiser to an arithmetic Frobenius then to an unramified π we get an unramified representation of the local Weil group, and then if we suddenly decide to change them, we get a new unramified representation, which may not be the same as the old one. One representation sends a geometric Frobenius F to (g, F), and the other sends it to $(g, F^{-1})^{-1} = (F(g^{-1}), F)$. Note that $F(g^{-1})$ is F-conjugate to g^{-1} (F-conjugacy classes are always made up of F-orbits because $g^{-1}g.F(g) = F(g)$), so in order to prove an "arithmetic" Buzzard-Gee conjecture assuming a "geometric" one, it would do to find some admissible map on L-groups which is inversion on a given maximal torus in the dual group, and which is an L-group homomorphism (that is, induces the identity on Galois groups).

9 Worked example: $\mathrm{GL}_2(F)$.

This is a bit of a lame example really. Let $A = M$ be the diagonal elements. Define $\delta(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)) = |a/b|$. Let N be the upper triangular matrices. Normalise Haar measures on $N(F)$, $M(F)$ and $G(F)$ so that the obvious maximal compact has measure 1. Recall that the Satake transform, denoted S, from $H(G, K)$ to $H(M(F), M(\mathcal{O}))$ is

$$(Sf)(m) = \delta(m)^{1/2} \int_N f(mn)\mathrm{d}n = \delta(m)^{-1/2} \int_N f(nm)\mathrm{d}n.$$

Let's try this with f the characteristic function of $K = \mathrm{GL}_2(\mathcal{O})$. If $m = \mathrm{diag}(a, b)$ then the only way $mn = \left(\begin{smallmatrix} a & an \\ 0 & b \end{smallmatrix}\right)$ can be in K is when a, b are units, and n is integral. So $Sf(m) = 0$ unless $m \in M(\mathcal{O})$, in which case a is a unit and hence n had better be an integer. We see that S sends the unit of $H(G, K)$ to the unit of $H(M(F), M(\mathcal{O}))$.

Next let's do the characteristic function of $K\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)K$. We need to see when $\left(\begin{smallmatrix} a & an \\ 0 & b \end{smallmatrix}\right)$ is in this; well, clearly a, an and b have to be integral, and $v(ab) = 1$, so there are two cases: either a is a uniformiser and b a unit, or vice-versa. Let's bash it out then. Sf is supported on these two cosets, that of $\mathrm{diag}(\varpi, 1)$ and $\mathrm{diag}(1, \varpi)$. We have

$$\begin{aligned}(Sf)(\mathrm{diag}(\varpi, 1)) &= q^{-1/2} \int_N f(\left(\begin{smallmatrix} \varpi & \varpi n \\ 0 & 1 \end{smallmatrix}\right))\mathrm{d}n \\ &= q^{-1/2} q = q^{1/2}\end{aligned}$$

and

$$\begin{aligned}(Sf)(\mathrm{diag}(1, \varpi)) &= q^{1/2} \int_N f(\left(\begin{smallmatrix} 1 & n \\ 0 & \varpi \end{smallmatrix}\right))\mathrm{d}n \\ &= q^{1/2}\end{aligned}$$

which has a certain pleasing symmetry about it, doesn't it. Aah—that's because we have to land in the Weyl group invariants!