Fourier transforms.

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1 Pontrjagin duality.

Noam Elkies wrote a brief note on this at
`http://www.math.harvard.edu/~elkies/M55b.99/pontrjagin.html`.

If G is a locally compact abelian group (LCAG) (that is, a **HAUSDORFF** topological abelian group such that any point has a neighbourhood contained in a compact set) then its dual is the continuous homomorphisms from G to the unit circle $\mathbf{T}$. Let $\hat{G}$ denote this abelian group. Any $g \in G$ gives (via evaluation) a homomorphism $\hat{G} \to \mathbf{T}$ and we put the weakest topology on $\hat{G}$ that makes all these homomorphisms continuous. The theorem is that with this topology $\hat{G}$ is an LCAG and the double-dual of G is canonically isomorphic with G as a topological group. In fact it gives an anti-equivalence of categories from the category of LCAGs (with continuous group homs as morphisms) to itself.

2 Examples.

The dual of a finite abelian group with the discrete topology is non-canonically isomorphic to itself.

The dual of $\mathbf{Z}$ is $\mathbf{T}$, and the dual of $\mathbf{T}$ is $\mathbf{Z}$.

More generally the dual of a compact group is discrete, and vice-versa.

The dual of $\mathbf{R}$ is (kind of non-canonically) $\mathbf{R}$ again, the map being that sending $r \in \mathbf{R}$ to the character $x \mapsto \exp(ixr)$ or $\exp(2\pi ixr)$ or whatever. The dual of $\mathbf{C}$ is kind of non-canonically $\mathbf{C}$ for the same reason.

If F is a finite extension of $\mathbf{Q}_p$ then the dual of F is non-canonically F; if one chooses a character $0 \neq \psi : F \to \mathbf{T}$ then one way of seeing the identification is that you can send $f \in F$ to the map $x \mapsto \psi(fx)$.

Similarly if R is the integers in F then the dual of the compact group R is non-canonically isomorphic to the discrete group F/R.

I think that the dual of a number field under addition (with the discrete topology) is related to the adeles of that field. That sounds wrong doesn't it. Is it that the dual of the multiplicative group is related to the ideles? I forget. Darn. This is in Tate's thesis but someone has my copy of Cassels-Froehlich.

3 Fourier transforms in some vast generality.

The idea is that if f is a reasonably well-behaved function on a LCAG G then its Fourier transform should be a simiarly reasonably well-behaved function on $\hat{G}$, thought of as the decomposition of f into characters.

4 Haar measure.

If G is any locally compact Hausdorff topological group then it has a left Haar measure. If G is furthermore abelian then this will equal the right Haar measure. This means that we can speak not only about continuous functions on G, but also measurable functions, and also L^p-functions, for $1 \leq p < \infty$, which are of course the measurable functions f for which the integral of $|f|^p$ converges, modulo the subspace of f for which the integral converges to zero. Functions in L^1 are referred to as "integrable". Note that because of the fact that $L^p(G)$ is a subquotient, rather than a subspace, of the measurable functions, there are in general no natural inclusions amongst these L^p spaces as p varies. All are Banach spaces though, the norm being the pth root of the integral. I *think* that if G is compact then "$L^p \subseteq L^1$ for $p > 1$". Note that (by convention) $L^\infty(G)$ is the "essentially bounded" functions f, that is, those that are bounded away from some set of measure zero, and the norm on L^∞ is of course the "essential supremum" of f.

Note however that in this generality one cannot speak about C^∞ functions on G, as far as I can see.

5 Formal definition in some vast generality.

If G is an LCAG, then G gets a Haar measure, defined up to a positive real. If now f is an L^1-function on G (that is, f is complex-valued, measurable, and the integral of $|f|$ over G converges) then its Fourier transform is the function $\hat{f} : \hat{G} \to \mathbf{C}$ sending χ to $\int_x f(x)\overline{\chi(x)}dx$. The theorem is that if we also put a Haar measure on $\hat{G}$ then $\hat{f}$ is bounded, continuous, and "vanishes at infinity", whatever that means. If f is in L^1 and $\hat{f}$ also happens to be in L^1 then it might be the case that one can recover f in L^1 as $f(x) = \int_\chi g(\chi)\chi(x)d\chi$ assuming one has picked the "correct" Haar measure on $\hat{G}$ (this is just a normalisation issue). I don't know a reference though.

One can also check that for G any LCAG, Fourier transform induces an isometric isomorphism $L^2(G) \to L^2(\hat{G})$. I read this in Cartan-Godement. I think this is "Plancherel's theorem"? Note that one has to be super-careful here by what one means by "induces": the Fourier transform converges for $f \in L^1(G)$ but if G isn't compact then there is no natural map $L^2(G) \to L^1(G)$ so we don't get a natural Fourier transform defined on all of $L^2(G)$ for free. What one does is firstly defines the Fourier transform on $L^1(G)$, and hence on $L^1(G) \cap L^2(G)$, and then extends by continuity to $L^2(G)$.

6 The reals.

Choose an identification $\mathbf{R} = \hat{\mathbf{R}}$. If f is C^∞ with compact support, or more generally in Schwartz space (which is C^∞ functions all of whose derivatives tend to zero faster than any power of $|x|$), then $\hat{f}$ is also in Schwartz space; in fact Fourier transform is an isomorphism from Schwarz space to itself. The transform of the convolution is the product of the transforms. One can read about this in Folland's "Real Analysis". The usual normalisation is that we use Lebesgue measure and set $\hat{f}(y) = \int f(x)e^{-2\pi ixy}dx$, so we identify $\mathbf{R}$ with its dual using that 2π factor.

The neat thing about these choices is that for $a > 0$ real, the Fourier transform of $x \mapsto \exp(-\pi ax^2)$ can be computed without too much trouble. If it is $g(y)$ then one checks by differentiating under the integral that $g'(y)$ is some integral involving what turns out to be $f'(x)$, because of our clever choice of f, and then by integration by parts we can get back to an integral of f again and deduce that $g'(y) = \frac{-2\pi y}{a}g(y)$, so $g(y) = ce^{-\pi y^2/a}$ for some constant c which is easily checked (set $y = 0$) to be $a^{-1/2}$. Note that the Fourier transform of g is now f again! But this is because f is even; usually when you do the Fourier transform twice to f you get $x \mapsto f(-x)$ (see Follard p251 for example).

If $f \in L^1(\mathbf{R})$ then $\hat{f}$ is a well-defined function on $\mathbf{R}$, and indeed it is uniformly continuous (see Baggett and Fulks, p105) and is zero iff $f = 0$ in L^1 (B&F p113). However, $\hat{f}$ might not be in L^1: the characteristic function of $[-1, 1]$ gives an easy counterexample; its Fourier transform is $\sin(x)/x$ (up to some constant) and this isn't in L^1. If however $f \in L^1$ and $\hat{f} \in L^1$ then f is continuous and $\hat{\hat{f}} = f(-x)$.

Plancherel's theorem is that if f is in L^1 and L^2 then $\hat{f}$ is in L^2 and the L^2-norms of f and $\hat{f}$ coincide.

7 The integers and the circle.

Reference: Chapter 8 of Folland.

Let $\mathbf{T}$ denote the circle group and say f is an L^1-function on $\mathbf{T}$. Its Fourier transform is in $\ell^\infty(\mathbf{Z})$, that is, a bounded sequence indexed by $\mathbf{Z}$, the nth term being $\int f(z)z^{-n}$, the integral with respect to a Haar measure, normalised so that the circle has measure 1. Note that boundedness of the Fourier coefficients is clear as indeed is the property that the ℓ^∞ norm of the coefficients is at most the L^1-norm of f. Note also that $f \in L^1(\mathbf{T})$ is determined by its Fourier transform $\hat{f} \in \ell^\infty(\mathbf{Z})$. A book called "Fourier Analysis" by Baggett and Fulks that I found in the Imperial library proves this on p15. This book also points out that not every bounded sequence arises in this way: indeed if f is in L^1 then there's a step function which is L^1-close to f and from this it's not hard to check that the bounded sequence in fact tends to zero in both directions (the Riemann-Lebesgue theorem). In fact, even a general element of $\ell^\infty(\mathbf{Z})$ which tends to zero in both directions may not be the Fourier transform of an element of $L^1(\mathbf{T})$: an example (from Baggett and Fulks, section 2.5) is given by the sequence defined by $c_n = 0$ for $-1 \leq n \leq 1$ and $c_n = \mathrm{sgn}(n)/log(|n|)$ for $|n| \geq 2$. No super-nice classification is known about the image of $L^1(\mathbf{T})$ although it's a fact that a sequence is in the image iff it's the convolution of two ℓ^2 sequences (p70 of B&F).

We can do much better if we assume $f \in L^2(\mathbf{T})$. In this case $\hat{f} \in \ell^2(\mathbf{Z})$ and the L^2-norm of f equals the ℓ^2-norm of $\hat{f}$. Equivalently, the characters $z \mapsto z^n$ for all n form an orthonormal basis of the Hilbert space $L^2(\mathbf{T})$, and the proof of this latter fact is essentially just the Stone-Weierstrass theorem (and the fact that the continuous functions are dense in L^2). As a consequence we see a special case of the general theorem that $L^2(G) = L^2(\hat{G})$: we have proved that $L^2(\mathbf{T})$ is isometrically isomorphic to $\ell^2(\mathbf{Z})$.

More generally if $1 \leq p \leq 2$ and $2 \leq q \leq \infty$ and $1/p + 1/q = 1$ then the Fourier transform takes $L^p(\mathbf{T})$ to $\ell^q(\mathbf{Z})$, and the Hausdorff-Young inequality is that the ℓ^q norm of $\hat{f}$ is $\leq$ the L^p-norm of f. We've just proved the cases $p = 1$ and $p = 2$ of this, and the general case now follows by some formal measure theory argument. This last is from Folland.

8 Schwarz space for totally disconnected topological spaces.

If X is a locally compact totally disconnected topological space then let $C_c^\infty(X)$ denote the vector space of locally constant complex-valued functions on X with compact support. One can make this space an algebra (a ring without a 1) by defining addition and multiplication pointwise and the spectrum of this ring (in some appropriate sense), with its Zariski topology, is X again. There is a sheaf of algebras on X and if you localise $C_c^\infty(X)$ at a prime ideal (all of which you get by fixing a point and evaluating functions at that point) then you recover the complex numbers.

9 A neat "duality" in the totally disconnected case.

If now G is a locally compact totally disconnected abelian group which is the union of its compact subgroups then $C_c^\infty(G)$ has a second multiplication if you fix a Haar measure: you can convolute. It turns out that Fourier transform induces an isomorphism of algebras between $C_c^\infty(G)$ (with convolution as multiplication) and $C_c^\infty(\hat{G})$ (with pointwise multiplication as multiplication). Is this some kind of analogue of the duality between Schwartz space and itself when $G = \mathbf{R}$?

Note also that for G locally compact totally disconnected abelian and the union of compact subgroups, there are bijections between $\hat{G}$ (in the sense above), the irreducible unitary reps of G (this much is kind of clear) and the smooth irreducible reps of G.

10 p-adic fields.

Let F be a finite extension of $\mathbf{Q}_p$. If we fix $\psi : F \to \mathbf{T}$ a non-trivial character then $\hat{F}$ becomes identified with F. I think that people Schwartz space in this setting to be the locally constant complex-valued functions on F with compact support. The remarks above should enable us to view functions on F under pointwise multiplication with functions on F under convolution. Let's fix ψ to be the usual thing: take trace from F to $\mathbf{Q}_p$ and then use the map sending $1/p^n$ to $\exp(2\pi i/p^n)$ for all n. Now F is identified with $\hat{F}$. Let π be a uniformiser for F and let's normalise Haar measure such that $\mathcal{O}$, the integers of F, have Haar measure 1. Note that ψ vanishes on D^{-1}, with D the different, and indeed D^{-1} will be the conductor of ψ.

For n an integer, let f_n denote the characteristic function of $\pi^n\mathcal{O}$. Then f_n has L^2-norm q^{-n} where q is the size of the residue field.

Let's compute the Fourier transform of f_n with respect to all our choices. Call it g_n. We have $g_n(y) = \int_{x\in\pi^n\mathcal{O}} \psi(-xy)dx$. Clearly this function vanishes outside $D^{-1}\pi^{-n}\mathcal{O}$ and is identically q^{-n} on it, so $g_n = q^{-n}f_{-c-n}$ with c the valuation of the different. Note that g_n has L^2-norm q^{c-n} so we've not quite done as well as Plancherel: we should have put in a normalising factor of $q^{-c/2}$ I guess.