Explicit Maass forms.

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1 Classical modular forms.

I'm going to do this in a bit of a kooky way.

If z is in the upper half plane and if n is a positive integer, then let me write $q_n(z) := e^{2\pi i n z}$ (most people just write q^n).

Lemma 1 (Key lemma!). *If p is prime then*

(a) $q_n(pz) = q_{np}(z)$.

(b) $\sum_{j=0}^{p-1} q_n\left(\frac{z+j}{p}\right) = pq_{n/p}(z)$ if $p \mid n$ and zero otherwise.

Proof. (a) is trivial. For (b) note that if $\zeta = e^{2\pi i/p}$ then the sum is $\sum_j \zeta^{nj} e^{2\pi i n z/p}$ which is zero for $p \nmid n$ and $pe^{2\pi i(n/p)z}$ for $p \mid n$. □

Now let N be a positive integer, let k be a positive integer, let χ be a Dirichlet character of level N with $\chi(-1) = (-1)^k$, and define a sequence $a_1, a_2, \ldots$ of complex numbers in the following way: $a_1 = 1$, a_p is anything you like, for p prime (except don't let them grow too quickly, say $|a_p| \leq 2p^{(k-1)/2}$ for all p), and then define a_{p^n} thus: if $p|N$ then $a_{p^n} = (a_p)^n$, and if $p \nmid n$ then define a_p by the usual recurrence relation:

$$\sum_{n=0}^{\infty} a_{p^n} X^n = (1 - a_p X + \chi(p) p^{k-1} X^2)^{-1}.$$

Now define a function on the upper half plane by $F(z) = \sum_{n \geq 1} a_n q_n(z)$.

Define an operator T_p on functions on the upper half plane: if $p \nmid N$ then define

$$(T_p f)(z) := \frac{1}{p} \sum_{i=0}^{p-1} f((z+i)/p) + p^{k-1} \chi(p) f(pz)$$

and if $p \mid N$ then drop the last term.

Lemma 2. *The function F above is an eigenvector for the Hecke operators T_p for all primes p, with eigenvalue a_p. Furthermore F is a holomorphic function on the upper half plane.*

Proof. The first sentence follows purely formally from the Key Lemma and the definition of the a_i. The second follows from the fact that a locally uniformly converging sum of holomorphic functions is holomorphic. □

Corollary 3. *"All that one has to do" to make sure that F is a weight k level N character χ cuspidal eigenform is to check boundedness at the cusps, and that it transforms in the appropriate manner under $\Gamma_0(N)$.*

Remark 4. This is sometimes not as hard as it seems! One can use converse theorems! We can define the L-function of F as $\sum_{n\geq 1} a_n/n^s$ and if this L-function and its twists are bounded in some reasonable sense, and satisfy some reasonable functional equations, then F *will* be a cusp form, the proof being that the functional equations and boundedness actually prove that F transforms correctly. Somehow the boundedness really is key (otherwise Artin's conjecture would be a theorem): this appears to be an observation of Hecke.

2 Maass forms.

There are some details here which are no doubt trivial to analysts but I had never seen them, so I've included them. For smooth reading they are best omitted!

Let $\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ denote the "non-Euclidean Laplacian" (a differential operator on the upper half plane, also known as the "Laplace-Beltrami operator").

Lemma 5. *If F is a smooth function on the upper half plane, and if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbf{R})$, considered as an automorphism of the upper half plane $z \mapsto (az+b)/(cz+d)$, then $\Delta(F \circ g) = (\Delta F) \circ g$.*

Proof. It suffices to check on a set of generators, for example $g = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and one just bashes it out in each case. ☐

Details (Definitely worth omitting! It's a tedious exercise.). *(a) $g = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. In this case we need to check that $\Delta(x+iy \mapsto F(x+iy+t))$ is $-y^2(F_{xx}(x+iy+t) + F_{yy}(x+iy+t))$ which is clear.*

(b) $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. This is trivial.

(c) $g = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In this case we have to check that $\Delta(x+iy \mapsto F(\lambda x + \lambda iy))$ is $-\lambda^2 y^2(F_{xx}(\lambda x + i\lambda y) + F_{yy}(\lambda x + i\lambda y))$. The left hand side is $-y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)F_(\lambda x + i\lambda y)$ and the two partial derivatives both float out a factor of λ^2 and we get the right answer.

(d) $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The only one left! In this case we have to check that $\Delta(x+iy \mapsto F(-x/(x^2+y^2) + iy/(x^2+y^2))$ is $-y^2/(x^2+y^2)^2(F_{xx}(-x/(x^2+y^2) + iy/(x^2+y^2)) + F_{yy}(-x/(x^2+y^2) + iy/(x^2+y^2)))$. So now I really have to understand the chain rule! Either that or I have to somehow be able to use the fact that $z \mapsto -1/z$ is holomorphic which would somehow help, I'm sure!

Consider the map $x+iy \mapsto F(-x/(x^2+y^2)+iy/(x^2+y^2))$. Let's compute the partial derivative of this map with respect to x. It's

$$\begin{aligned} x+iy \mapsto & (-(x^2+y^2)+2x^2)/(x^2+y^2)^2 F_x(-x/(x^2+y^2)+iy/(x^2+y^2)) \\ & - 2xy/(x^2+y^2)^2 F_y(-x/(x^2+y^2)+iy/(x^2+y^2)) \\ = & \frac{x^2-y^2}{(x^2+y^2)^2} F_x(-x/(x^2+y^2)+iy/(x^2+y^2)) \\ & - 2xy/(x^2+y^2)^2 F_y(-x/(x^2+y^2)+iy/(x^2+y^2)). \end{aligned}$$

Now let's compute the partial derivative of this *with respect to x! It's*

$$
\begin{aligned}
&\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)^2 F_{xx}(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&-2\frac{(x^2-y^2)(2xy)}{(x^2+y^2)^4}F_{xy}(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+\frac{2x(x^2+y^2)-4x(x^2-y^2)}{(x^2+y^2)^3}F_x(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+(2xy/(x^2+y^2)^2)^2F_{yy}(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+(-2y(x^2+y^2)+4xy(2x))/(x^2+y^2)^3F_y(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&=\frac{(x^2-y^2)^2}{(x^2+y^2)^4}F_{xx}(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&-\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}F_{xy}(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+\frac{4x^2y^2}{(x^2+y^2)^4}F_{yy}(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+\frac{-2x^3+6xy^2}{(x^2+y^2)^3}F_x(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+\frac{6yx^2-2y^3}{(x^2+y^2)^3}F_y(-x/(x^2+y^2)+iy/(x^2+y^2))
\end{aligned}
$$

We now have to play the same game with y: the first derivative is

$$
\begin{aligned}
x+iy\mapsto &\frac{2xy}{(x^2+y^2)^2}F_x(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+\frac{x^2-y^2}{(x^2+y^2)^2}F_y(-x/(x^2+y^2)+iy/(x^2+y^2))
\end{aligned}
$$

and the second is

$$
\begin{aligned}
&\frac{4x^2y^2}{(x^2+y^2)^4}F_{xx}(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}F_{xy}(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+\frac{(x^2-y^2)^2}{(x^2+y^2)^4}F_{yy}(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+\frac{2x^3-6xy^2}{(x^2+y^2)^3}F_x(-x/(x^2+y^2)+iy/(x^2+y^2))\\
&+\frac{-6yx^2+2y^3}{(x^2+y^2)^3}F_y(-x/(x^2+y^2)+iy/(x^2+y^2))
\end{aligned}
$$

and the sum is, slightly too miraculously,

$$
\frac{1}{(x^2+y^2)^2}(F_{xx}+F_{yy})(-x/(x^2+y^2)+iy/(x^2+y^2))
$$

which is what we wanted to prove.

Now let ν be a complex number (my understanding is that for cuspidal Maass forms the positivity of a certain operator will imply that $1/4-\nu^2$ is a positive real, but we never need to assume this in the "local" theory at infinity).

Define the K-Bessel function $K_\nu(y)$, a function from the positive reals to the complexes, by

$$K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^\nu dt/t.$$

This integral is absolutely convergent no problems. Furthermore pari can compute it efficiently and quickly (it's the function `besselk(nu,y)`) but the implementation appears to be broken in the current version (2.4.1) when $\nu = 0$, which is a case of definite interest—the workaround is to use $\nu = (1e - 30) * I$ instead!). This is a certain kind of Bessel function; Bessel functions usually oscillate, there are usually two around, and they're the solutions to a second order differential equation, kind of analogous to sin and cos, but we're not using those ones, we're using a certain linear combination of them that decreases exponentially as y grows, a kind of analogue of e^{-t}; the other "natural" basis function for the solution of the differential equation has exponential growth as y tends to infinity (but note that $K_\nu(y)$ blows up exponentially as y tends to zero). Anyway, here's the standard fact about Bessel functions:

Lemma 6. *K_ν is a solution to the differential equation*

$$y^2 d^2F/dy^2 + y dF/dy - (y^2 + \nu^2)F = 0.$$

Proof. Differentiate under the integral (and multiply by 8); one now has to check that for any positive real number y we have

$$\int_{t=0}^\infty (t^2y^2 - 2y^2 + y^2/t^2 - 2yt - 2y/t - 4\nu^2) e^{-y(t+t^{-1})/2} t^\nu dt/t = 0$$

and we're going to do this by spotting that the integrand is the derivative (with respect to t) of $2(-yt - 2\nu + y/t)e^{-y(t+1/t)/2}t^{\nu-1}$ and that (for positive real y) as t tends either to 0 or infinity this function is exponentially decreasing to zero. $\square$

Corollary 7. *The functions on the upper half plane defined by $x + iy \mapsto y^{1/2}K_\nu(2\pi y)e^{2\pi ix}$ and $x + iy \mapsto y^{1/2}K_\nu(2\pi y)e^{-2\pi ix}$ are both eigenvectors for Δ, with eigenvalue $(1/4 - \nu^2)$.*

Proof. Just bash it out; each function is a product of a function of x only and a function of y only, so it's easy. $\square$

Details. *Let's do the first function. Explicitly, applying Δ gives us the function*

$$x + iy \mapsto -y^2(-4\pi^2 y^{1/2}K_\nu(2\pi y) - 4^{-1}y^{-3/2}K_\nu(2\pi y) + y^{-1/2}2\pi K_\nu'(2\pi y) + y^{1/2}4\pi^2 K_\nu''(2\pi y))e^{2\pi ix}$$

which simplifies to

$$y^{1/2}(4y^2\pi^2 K_\nu(2\pi y) + 4^{-1}K_\nu(2\pi y) - 2\pi y K_\nu'(2\pi y) - 4\pi^2 y^2 K_\nu''(2\pi y))e^{2\pi ix}$$

so it suffices to check that

$$(1/4 - \nu^2)K_\nu(2\pi y) = 4y^2\pi^2 K_\nu(2\pi y) + (1/4)K_\nu(2\pi y) - 2\pi y K_\nu'(2\pi y) - 4\pi^2 y^2 K_\nu''(2\pi y),$$

that is, that

$$(\nu^2 + 4y^2\pi^2)K_\nu(2\pi y) - 2\pi y K_\nu'(2\pi y) - 4\pi^2 y^2 K_\nu''(2\pi y) = 0,$$

and it does.

For the second function, the details are essentially the same.

Now choose a sign $\epsilon \in \{+1, -1\}$ (this is going to be the "sign" of the Maass form: this has nothing to do with Atkin-Lehner operators, this is all to do with the infinity type or (from the Galois point of view) whether complex conjucation is $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$. Define a function $W^\uparrow$ on the upper half plane by

$$W^\uparrow(x + iy) = y^{1/2}K_\nu(2\pi y)\exp(2\pi ix)$$

and define a function $W^\downarrow$ on the lower half plane by

$$W^\downarrow(x+iy) = (-y)^{1/2}K_\nu(2\pi(-y))\exp(2\pi ix)$$

(so $W^\downarrow(z) = W^\uparrow(\overline{z})$).

Now for n a positive real number, define functions on the upper half plane thus:

$$Q_n^+(z) = (W^\uparrow(nz) + W^\downarrow(-nz))$$

and

$$Q_n^-(z) = (W^\uparrow(nz) - W^\downarrow(-nz)).$$

Lemma 8. *Q_n^+ and Q_n^- are eigenvectors for Δ with eigenvalue $(1/4-\nu^2)$.*

Proof. The previous corollory showed that $W^\uparrow$ was, and from Lemma 5 (the invariance under $\mathrm{GL}_2^+(\mathbf{R})$ we deduce that $z \mapsto W(nz)$ will be for any positive real n. Similarly the previous corollary showed that $W^\downarrow(-z)$ (a function on the upperr half plane) was, and now again by invariance of the differential operator we're done. □

Now let us restrict to n a positive integer (because we want our Maass forms to satisfy $F(z) = F(z+1)$).

Lemma 9. *(Key Lemma for Maass forms) If p is prime then*

(a) $Q_n^+(pz) = Q_{np}^+(z)$ and similarly for Q^-, and

(b) $\sum_{j=0}^{p-1} Q_n^+\left(\frac{z+j}{p}\right) = pQ_{n/p}^+(z)$ if $p \mid n$ and zero otherwise, and similarly for Q^-.

Proof. It suffices to prove these assertions with $Q_n^+(z)$ replaced by $W(nz)$ and $W(-nz)$. Now (a) is trivial, and for (b) note that if $\zeta = e^{2\pi i/p}$ then the sum is $\sum_j \zeta^{nj}W(nz/p)$ which is zero for $p \nmid n$ and $pW(nz/p)$ for $p \mid n$. □

You can see what's coming now, right?

Now let N be a positive integer, let χ be an even Dirichlet character of level N, let ϵ be a sign, think of k as zero, and define a sequence $b_1, b_2, \ldots$ of complex numbers in the following way: $b_1 = 1$, b_p is anything you like with $|b_p| \leq 2p^{-1/2}$, and define $b_{p^n} = (b_p)^n$ if $p \mid N$ and via the recurrence

$$\sum_{n=0}^{\infty} b_{p^n}X^n = (1 - b_pX + \chi(p)p^{-1}X^2)^{-1}$$

if $p \nmid N$. Now define a function on the upper half plane by $F(z) = \sum_{n\geq 1} b_nQ_n^\epsilon(z)$. The operator T_p is defined as in the previous section, and we see

Lemma 10. *The function F above is an eigenvector for the Hecke operators T_p for all primes p, with eigenvalue b_p, and furthermore F is an eigenvector for Δ with eigenvalue $(1/4-\nu^2)$.*

Proof. The combinatorics is the same as before and convergence is easy and a sum of eigenvectors (with the same eigenvalue) is an eigenvector. □

Corollary 11. *"All that one has to do" to make sure that F is a Maass form is to check boundedness at the cusps, and that F transforms in the appropriate manner under $\Gamma_0(N)$.*

Remark 12. And of course one can use converse theorems, who don't care whether you're dealing with holomorphic or non-holomorphic forms, to do this. Note that the L-function of F above is in fact $\sum_{n\geq 1} b_n\sqrt{n}/n^s$; this square root had to appear somewhere, because the analogy of Maass forms with modular forms isn't perfect: a Maass form looks a bit like a non-holomorphic form of "weight 0" (for the purposes of the action of $\Gamma_1(N)$ but also a bit like a non-holomorphic form of "weight 1" (for the purposes of Galois representations). For the purposes of notation I'm going to set $a_n = b_n\sqrt{n}$. In the algebraic Maass form case it's the a_n which will be the traces of Frobenius, even though the b_n are the Hecke eigenvalues; this is a funny twist which one doesn't see in the holomorphic case, and it's due to the fact that Maass forms can't quite make up their mind as to whether they are "weight 0" or "weight 1".

3 Explicit examples.

3.1 An algebraic example.

I can write down an example of a Maass form which gives rise to a Galois representation! This was a bit of a pain for reasons I'll explain below, but because computing 200,000 Q-expansion coefficients isn't much trouble in 2007 I could still do it.

Before I start, let me explain something about why I chose this particular representation. If I really want pari to compute a sum of Bessel functions and then attempt to check that it's invariant under some subgroup $\Gamma_1(N)$ then it's nice to have N as small as possible, because for z near the real line, to compute the sum accurately one need to sum hundreds of thousands of Bessel functions, and the bigger N is, the closer to the real line one of $z, \gamma z$ is going to be for $\gamma \in \Gamma_0(N)$ with $c \neq 0$. Fortunately my "dihedral forms" script that I wrote when computing weight 1 holomorphic cusp forms apparently computed all 2-dimenensional complex Galois representations that were induced from characters on quadratic fields, both even or odd, so I could use that script and search to find even representations. The actual magma script is commented out in the TeX file.

Using this strategy I found continuous even 2-dimensional complex representations of conductors 136, 145, 148, 205 and 221. Now 136 is a multiple of 8 so I went for 145 (which is both odd and squarefree); let me explain this example carefully.

The class group of $K := \mathbf{Q}(\sqrt{145})$ is cyclic of order 4 and totally real. Let H denote the associated Hilbert class field. According to pari, H is the splitting field of $f := x^4 - x^3 - 3x^2 + x + 1$, a polynomial whose discriminant is $5^2.29$. The primes 5 and 29 are the only ramified primes in K. Now $\mathrm{Gal}(H/\mathbf{Q}) = D_8$, the dihedral group with 8 elements, and this group has an irreducible 2-dimensional complex representation. The only elements with non-zero traces are the identity (trace 2) and the non-trivial central element (trace -2). The non-trivial central element generates a subgroup of order 2 which fixes a field of degree 4 over $\mathbf{Q}$ and this field is $\mathbf{Q}(\sqrt{5}, \sqrt{29})$. The determinant of the representation is non-trivial (think of D_8 acting on a square; there are flips) and the determinant cuts out a quadratic extension of $\mathbf{Q}$ which is just K. Hence the character of our Maass form is going to be $(\cdot/145) = (\cdot/5)(\cdot/29)$ and for p a prime with $p \notin \{5, 29\}$ we have $a_p = 0$ unless p splits completely in $\mathbf{Q}(\sqrt{5}, \sqrt{29})$.

Non-zero a_p are rather sporadic and take a while to start (for the usual reasons). Magma tells me that $a_5 = a_{29} = -1$ (this is a brute force calculation of Frobenius acting on inertial invariants, which it can do because magma can see the ray class character which is giving rise to the form) and all the other a_p are zero for $p < 59$. We have $a_{59} = a_{71} = -2, a_{109} = 2$ and $a_{139} = 2$ and those are the only non-zero a_p up to 139. The representation is induced from all three of $\mathbf{Q}(\sqrt{29}), \mathbf{Q}(\sqrt{5})$ and $\mathbf{Q}(\sqrt{145})$!

The automorphic representation has non-trivial character at 5 and 29 and hence is ramified principal series at these primes. Magma tells me that $e = f = g = 2$ at 5 and 29, so in each case inertia is order 2 and must be acting in each case as something with order 2 (because it corresponds to a tamely ramified principal series with one unramified character and one quadratic tamely ramified character).

Magma has done the dirty work of computing a_5 and a_{29}; we need to know a_p for much much larger p if we want convergence, but this is easy. For a good prime, its trace will be zero unless it splits completely in $\mathbf{Q}(\sqrt{5}, \sqrt{29})$. If p splits completely in this extension (i.e.,if both 5 and 29 are squares mod p) then a_p will be $+2$ or -2, and it'll be $+2$ iff p splits completely in H, which one can test by factoring f mod p. The pari code is commented out in the TeX file. Note that the a_ps above are traces, so they're not the b_p in the Q-expansion, there's a factor of $\sqrt{p}$ missing, as I explained earlier.

Using these observations, I could easily write a pari script that firstly computed a_n for all $n < 200000$ (in a few seconds) and then actually summed the infinite sum that gives me a Maass form. The script is commented out in the TeX file, as is a sample run, which indicates that the infinite sum does appear to be stable under $\Gamma_1(145)$.

I have pari files containing other algebraic examples, for example one involving the class group of $K := \mathbf{Q}(\sqrt{79})$ (which is cyclic of order 3 over K and gives a totally real S_3-extension).

3.2 A non-algebraic example.

It's much easier to come up with non-algebraic examples, from a computational point of view, because one can work with much much lower level: the Grossencharacter on the quadratic field doesn't have to have trivial infinity type so one can use quadratic fields of very small discriminant and Grossencharacters of conductor 1. Let's do the one in Gelbart's book, the one with infinity type π_s^+ with $s = 2\pi i/\log(\sqrt{2}-1)$ and Δ-eigenvalue $(1-s^2)/4$, so $\nu = \pi i/\log(\sqrt{2}-1)$.

Here's the mathematics. Set $L = \mathbf{Q}(\sqrt{2})$. We want a Grossencharacter on L and $\mathbf{A}_L^\times$ is $L^\times.\hat{\mathcal{O}_L}^\times.(\mathbf{R}_{>0})^2$ because L has class number 1 and a fundamental unit of norm -1. Hence a Grossencharacter of conductor 1 can be thought of as a function on $(\mathbf{R}_{>0})^2/\Gamma$ with Γ the totally positive units embedded diagonally. We want the Grossencharacter to be unitary, so it will be of the form $(x_1, x_2) \mapsto x_1^{it_1} x_2^{it_2}$ with x_1 corresponding to the "obvious" embedding $\sqrt{2} > 0$, say. We want the character to be trivial on $3+2\sqrt{2}$ so $(3+2\sqrt{2})^{it_1 - it_2} = 1$ and hence $i(t_1-t_2)\log(3+2\sqrt{2}) = 2\pi i n$ for some integer n and hence $t_1 - t_2 = \pi n/\log(1+\sqrt{2})$ for some integer n and Gelbart chooses $n = 2$ for some reason and furthermore sets $t_1 + t_2 = 0$, so $t_1 = \pi/\log(1+\sqrt{2}) = 3.5644279\ldots$ and $t_2 = -t_1$.

This defines a grossencharacter χ. To compute χ of a uniformiser at some prime $\mathfrak{p}$ of L, choose a totally positive generator λ of $\mathfrak{p}$ and then embed this generator into $(\mathbf{R}_{>0})^2$ in the obvious way and evaluate. I don't have to worry about arithmetic vs geometric Frobenii because when I induce the answer will be independent of my choice. Furthermore the nice thing about choosing $n = 2$ above rather than 1 is that you don't need a totally positive generator; any generator will do, and you embed it into $(\mathbf{R}_{>0})^2$ by just taking the absolute values of both embeddings.

The idea of course is to consider the representation of the Weil group of E induced by this Grossencharacter, and then induce up to the Weil group of $\mathbf{Q}$ and get a representation to $\mathrm{GL}_2(\mathbf{C})$. Again I'll work with a_p, the traces of Frobenius, rather than b_p, the Q-expansion coefficients (and hence the Hecke eigenvalues). Here's what's going on explicitly. For p odd the automorphic representation will be unramified principal series; if p is inert in $\mathbf{Q}(\sqrt{2})$ then $a_p = 0$ and if $p = \mathfrak{p}\overline{\mathfrak{p}}$ is split then $a_p = \chi(\mathfrak{p}) + \chi(\overline{\mathfrak{p}})$. Computing at 2 is a special case: the Grossencharacter is unramified there, and in fact it's trivial there because $\chi(\sqrt{2}) = |\sqrt{2}|^{it_1}|\sqrt{2}|^{it_2} = 1$. So locally the Weil representation looks like the induction from $\mathbf{Q}_2(\sqrt{2})$ to $\mathbf{Q}_2$ of the trivial representation, and this is just trivial plus the character of order 2 corresponding to the quadratic extension; hence the Weil representation is reducible, and the smooth admissible representation is principal series with one ramified (of conductor 8) and one unramified character. Hence the caracter of the Maass form will be $\chi(n) := (2/n)$ and the level will be 8 (Gelbart says it's 2 in his book but I think he's wrong). Furthermore we have $a_2 = 1$ (the trace of Frobenius on the inertial invariants). The a_p will be zero if p is 3 or 5 mod 8, and for $p = 1$ or 7 mod 8 write $p = (a+b\sqrt{2})(c+d\sqrt{d})$ and then compute $z = |a+b\sqrt{2}|^{it_1}|a-b\sqrt{2}|^{it_2}$ (a complex number of norm 1) and set $a_p = z + \overline{z}$. This really works: again I have a pari script that computes only thousands of Q-expansion coefficients and gives a function that looks computationally invariant under $\Gamma_1(8)$.

As an example, we have $a_7 = z + \overline{z}$ with $z = (3+\sqrt{2})^{it_1}(3-\sqrt{2})^{-it_1}$ with, recall, $t_1 = \pi/\log(1+\sqrt{2})$, so $a_7 = -1.747936377\ldots$. Explicitly, we have $a_7 = 2\cos(\pi r)$ with

$$r = \frac{\log(3+\sqrt{2}) - \log(3-\sqrt{2})}{\log(1+\sqrt{2})}$$

and one imagines that this is wholly transcendental! On the other hand we have $a_2 = 1$, so it's hard to imagine that any twist of this form will be algebraic in any way.

4 Some general theory of Maass forms.

I wrote this up at some point; it doesn't really go with the rest of this note but I'll put it here anyway.

If $\Gamma = \Gamma_1(N)$ then a *Maass form* is a C^∞ function f on the upper half plane such that $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$, and such that firstly $\Delta f = \lambda f$ for some $\lambda \in \mathbf{C}$, with Δ the differential operator $-y^2(\partial^2/\partial^2 x+\partial^2/\partial^2 y)$, and secondly a boundedness condition: there is an N such that $(f\circ\gamma)(x+iy)=O(y^N)$ for all $\gamma\in \mathrm{SL}_2(\mathbf{Z})$. We say furthermore that f is a cusp form if $\int_0^1(f\circ\gamma)(z+x)dx=0$ for all $\gamma\in\mathrm{SL}_2(\mathbf{Z})$.

Example: if $\Re(s)>1$ then the non-holomorphic Eisenstein series $E_s(\tau)=\sum'_{m,n}\Im(z)^s/|m\tau+n|^{2s}$ works, because Δ is $\mathrm{GL}_2^+(\mathbf{R})$-invariant and the map from the upper half plane to the reals sending z to $\Im(z)^s$ is an eigenvector for it with eigenvalue $s(s-1)$, and E_s is just a sum of such things translated around by $\mathrm{SL}_2(\mathbf{Z})$ so it's also an eigenvector with eigenvalue $s(s-1)$.

The functions E_s extend meromorphically to all $s\in\mathbf{C}$, by the way, with poles at $s=1$ and $s=0$ and no other poles. We have $E_s(z+1)=E_s(z)$ of course, so we can write $E_s(z)=\sum_{r\in\mathbf{Z}}a_{s,r}(y)e^{2\pi irx}$ with $a_{s,r}(y)$ some function on the reals. It turns out that $a_{s,0}(y)$ is some linear combination of y^s and y^{1-s}, and $a_{s,r}(y)$ for $r\neq 0$ is some constant times $y^{1/2}$ times a Bessel function $y\mapsto K_{s-1/2}(2\pi|r|y)$. In fact once you have these facts it's not difficult to analytically continue E_s, because $|K_s(y)|\leq Ce^{-y/2}$ for some constant $C=C(s)$.

More generally, for a level 1 Maass form we have $f(z+1)=f(z)$ and hence we can write $f(z)=\sum_{r=-\infty}^{\infty}a_r(y)e^{2\pi irx}$ and if $\Delta f=(1/4-\nu^2)f$ then it's shown in Bump's book that $a_r(y)$ must be a constant C_r times $\sqrt{y}K_v(2\pi|r|y)$, the point being that one can check that $a_r(y)/\sqrt{y}$ satisfies a certain second order differential equation, which has two linearly independent solutions, one of which has exponential growth and the other of which is Bessel's function.

Here's a mad involution: the function on the upper half plane sending $x+iy$ to $-x+iy$ sends Maass forms to Maass forms and preserves Δ-eigenvalues. So the space of Maass forms can be broken up into a direct sum of two subspaces, the odd ones and the even ones. For the even ones we have $C_r=C_{-r}$ and for the odd ones we have $C_{-r}=-C_r$. For a cusp form, by the way, we have $C_0=0$.

The L-function of a Maass form is just $\sum_{r>0}C_r/r^s$. One can check without too much trouble that if f is a cusp form then $C_r=O(r^{1/2})$ and hence the L-function converges for $\Re(s)>3/2$. In fact Bump says that the Rankin-Selberg method shows that for a cusp form, the L-function converges for $\Re(s)>1$ (but remember what he said about the Ramanujan conjecture :-/)

If f is a cuspidal Maass form with Δ-eigenvalue $(1/4-\nu^2)$, and if we set $\epsilon=0$ if f is even and $\epsilon=-1$ if f is odd, then we can define the completed L-function of f thus:

$$\Lambda(s)=\pi^{-s}\Gamma((s+\epsilon+\nu)/2)\Gamma((s+\epsilon-\nu)/2)L(s)$$

and the theorem is that

$$\Lambda(s)=(-1)^{\epsilon}\Lambda(1-s).$$

This is only for level 1 eigenforms, unfortunately.

More random facts. Δ only has positive real eigenvalues on the cusp forms, so if $(1/4-\nu^2)$ is an eigenvalue then ν is either real and in $(-1/2,1/2)$, or pure imaginary. Selberg proved that the first case does not arise for $\mathrm{SL}_2(\mathbf{Z})$ and conjectured that it wouldn't arise for congruence subgroups (but for an arbitrary discrete group Γ, non-zero real values of ν can occur).

If Γ comes from a non-split indefinite quaternion algebra then the eigenvalues of Δ can be ordered $0=\lambda_0<\lambda 1<\ldots$. The zero comes from the constant function, which is a cusp form as there are no cusps. A consequence of the Selberg trace formula is "Weyl's law", that the number of eigenvalues λ_i with $\lambda_i<N$ is some non-zero constant (related to the volume of the quotient of the upper half plane) times N, so in particular there are eigenvalues. But if Γ has cusps, then we have the Eisenstein series, and also the discrete spectrum, which is the constant functions and the cusp forms (and Weyl's law still holds). Note that Maass forms coming from Grossencharacters over quadratic extensions of $\mathbf{Q}$ are another way to prove the existence of Maass forms, but one can count eigenvalues and check that "almost no" Maass forms are of this form.