Explicit formulae for the coeffts of the char power series of U

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Last modified 01/04/2006. Note that David Roe proved the observation in the case $p = 3$ shortly after.

1 Introduction

Harvard, 30/3/06. Frank Calegari just explained the following thing to me. We know that if $\sum_{n\geq 0} a_n T^n$ is the char power series of U acting on tame level N overconvergent modular forms of weight $\mathcal{W}_i$, with $\mathcal{W}_i$ a connected component of weight space, then $a_1 = 1$ and $a_n \in \mathbf{Z}_p[[w]]$ for all $n \geq 1$. This means the following. If we had, vaguely speaking, some "uniform" lower bounds for the char power series near the boundary of weight space, then one could really say what was going on there in the following sense: for $w \in \mathcal{W}_i$, thought of as the open unit disc, then sufficiently near the boundary we have $|a_n(w)| = |w^t|$ where t is the valuation of a_n mod p in $\mathbf{F}_p[[w]]$. Furthermore because of these "uniform lower bounds" (which I think just boil down to something like lower bounds for these valuations for n suff large) we actually get information about the first few slopes of the char power series of U near the boundary! So computing these valuations is somehow important when it comes to understanding the boundary. Note also, however, that it seems to tell you nothing about components of the eigencurve that live entirely in the region where the slope is ≥ 1—perhaps one conjectures that these do not exist, but if they do then this method will never tell you anything about them.

2 The trace formula.

Let H be pari's *qfbhclassno* function; so $H(n) = 0$ for $n < 0$, $H(0) = -1/12$, and for $n > 0$, $H(n)$ is the number of equivalence classes of positive definite binary quadratic forms with discriminant $-n$, counted with certain multiplicities.

For $k \geq 4$ even, $m \geq 1$, and $t \in \mathbf{Z}$ with $t^2 \leq 4m$, define $P_k(t, m) = \frac{\rho^{k-1}-\overline{\rho}^{k-1}}{\rho-\overline{\rho}}$, where ρ and $\overline{\rho}$ satisfy $\rho + \overline{\rho} = t$ and $\rho\overline{\rho} = m$. Note that $P_k(t, m) = P_k(-t, m)$ as k is even.

Theorem 2.1. *The trace of $T(m)$ on $S_k(\mathrm{SL}_2(\mathbf{Z}))$ is*

$$-\left(\frac{1}{2}\sum_{t\in\mathbf{Z}} P_k(t, m)H(4m - t^2)\right) - \frac{1}{2}\sum_{dd'=m} \min\{d, d'\}^{k-1}.$$

Now say $m = p^n$. To compute the trace of $(U_p)^m$ on overconvergent forms of weight k and level 1 we let k_n tend to k in weight space and to ∞ in the reals, and take the limit of the traces. Before we do this, let's make some simplifications. If $p|t$ then both ρ and $\overline{\rho}$ are roots of $X^2 = 0$ mod p, so both have positive valuation, and the sum disappears. If $p \nmid t$ then precisely one of the roots (say $\overline{\rho}$) has positive valuation, so that doesn't affect us. We deduce

Corollary 2.2. *The trace of $(U_p)^n$ on cuspidal overconvergent forms of tame level 1 and weight $k \in 2\mathbf{Z}$ is*

$$-\sum_{t\geq 1, p\nmid t} \frac{\rho^{k-1}}{\rho - \overline{\rho}} H(4p^n - t^2) - 1.$$

where here the sum is only over t with $t^2 \leq 4p^n$ and ρ is the unit root of $X^2 - tX + p^n$ and $\overline{\rho}$ is the other root.

Example: Let $\sqrt{-7}$ be the square root of -7 in $\mathbf{Z}_2$ which is 1 mod 4 and let ρ_7 be $\frac{1+\sqrt{-7}}{2}$. Then the trace of U_2 is

$$-\frac{\rho_7^{k-1}}{\sqrt{-7}} - 1 = -\frac{\rho_7^k}{\rho_7\sqrt{-7}} - 1$$

and so a_1 is going to be minus this. Now if $w = 5^k - 1$ then $\rho_7^k = 5^{\lambda k} = (1+w)^\lambda =$ where $\lambda \in \mathbf{Z}_2$ satisfies $5^\lambda = -\rho_7$ (the minus sign is because k is even and ρ_7 is 3 mod 4) so $\lambda = \log(-\rho_7)/\log(5) = 1 + 2^3 + 2^4 + 2^5 + 2^7 + 2^9 + \cdots$. In particular, the coefficient of T in the char power series of U_2 on overconvergent forms of weight 2 is $a_1(w)$, where $a_1 \in \mathbf{Z}_2[[w]]$ and, modulo 2, $a_1(w) = (1+w)^{(1+2^3+\cdots)} - 1 = w + \ldots$ and we deduce that for $|w| > 1/2$ the smallest slope of U_2 is at most $v(w)$ (but of course Buzzard-Kilford tells you that it is exactly $v(w)$, even for $|w| > 1/8$, which checks out because if you expand out $a_1(w)$ you get $8 + w + 12w^2 + 12 * w^3 + 2 * w^4 + \cdots$ modulo 16. Remark: the trace vanishes at weight $w = 2^3+2^5+2^6+2^7+2^8+2^{13}+2^{16}+2^{18}+2^{19}+\cdots$, and this corresponds to $k = 2+2^2+2^3+2^{11}+2^{15}+2^{16}+2^{18}+\cdots$, which, unsurprisingly, is close to 14.

Example: to compute the trace of U_4 we need to sum over $t \geq 1$, t odd, $t^2 \leq 16$, so $t = 1$ and $t = 3$ come into the picture. The roots we get are $\frac{t\pm\sqrt{t^2-16}}{2}$ so if we normalise our square roots so that they are always 1 mod 4 then the two quadratic surds that come into play are $-\rho_7^2 = \frac{3-\sqrt{-7}}{2}$ and $\rho_{15} = \frac{1+\sqrt{-15}}{2}$. Explicitly, the trace of U_4 is

$$-2\frac{\rho_{15}^{k-1}}{\sqrt{-15}} - \frac{(-\rho_7^2)^{k-1}}{-\sqrt{-7}} - 1$$

and this means that the coefficient of T^2 in the char power series of U_2 on weight k forms (k even) is (after some work and a bit of cancellation)

$$1 + \frac{\rho_7^k}{\rho_7\sqrt{-7}} + \frac{\rho_7^{2k}}{-7\rho_7} + \frac{\rho_{15}^k}{\rho_{15}\sqrt{-15}}.$$

One wonders whether the miraculous cancellation that just occurred (no 2s in denominator) is a general fact.

3 Computations.

For a power series $a = \sum c_i W^i \in \mathbf{Z}_p[[W]]$, define $\operatorname{ord}(a)$ to be the smallest $i \geq 0$ such that the coefficient of W^i is a unit mod p. Better: for $p > 2$ define $\operatorname{nord}(a)$ to be the minimum of $i + v(c_i)$, $i \geq 0$, and for $p = 2$ define $\operatorname{nord}(a)$ to be the minimum of $i + \frac{1}{3}v(c_i)$. The point of this definition is that if $\operatorname{nord}(a) = j$ then for $1 > |w| > 1/p$ (resp. $1 > |w| > 1/8$) we know that $|a(w)| = |w^j|$.

3.1 $p = 2$

Nothing new to say here: Buzzard-Kilford tells you the answer near the boundary anyway.

n	$\text{nord}(a_n)$
0	0
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55
11	66
12	78

So the first few "boundary slopes" look like 0,1,2,3,4,5,6,7,8,9,10,11,12. Of course this alone isn't a proof of this, because a_{13} might perhaps have $\text{nord}(a_{13}) = 1$ or something.

3.2 $p = 3$.

n	$\text{nord}(a_n)$
0	0
1	2
2	6
3	12
4	20
5	30
6	42
7	56
8	72
9	90

So the first few boundary slopes here look like 2,4,6,8,10,12,14,16,18. There is a theorem waiting to be proved here. Because I used nord rather than Frank's ord, this really is a strong indication that for $|w| > 1/3$ the slopes of the newton poly of U_3 near the boundary are $v(w)$ times $2, 4, 6, 8, 10, \ldots$.

3.3 $p = 5$.

Here there are two components of weight space where something is happening, and slightly different things are happening on each component. This phenomenon was noticed by Kilford when doing explicit computations with $|w| = 1/4$ for overconvergent forms of integral weight. The results here are an indication that the obvious generalisation of what he says may well be true: the slopes he gets at $|w| = 1/4$ are telling you what's happening as one approaches the boundary.

The weight 0 component:

n	nord(a_n)
0	0
1	1
2	4
3	8
4	14
5	22
6	31
7	42
8	54
9	68
10	84

so the first few slopes are perhaps $v(w)$ times 1,3,4,6,8,9,11,12,14,16. Kilford (or possibly Calegari earlier) observed that the ith term in this sequence might be $\lfloor\frac{8i}{5}\rfloor$.

The weight 2 component:

n	nord(a_n)
0	0
1	2
2	6
3	11
4	18
5	26
6	36
7	48
8	61
9	76
10	92

so the slopes are 2,4,5,7,8,10,12,13,15,16. Again I see in Kilford's paper that the ith term in this sequence might be $\lfloor\frac{8i+4}{5}\rfloor$.

3.4 $p = 7$

One can go on and on. For $p = 7$ the slopes appear to be 1,2,3,5,6,7,9,10,... in the identity component. Does anyone have a general conjecture?

3.5 Notes on the program.

One can go on. Note that the program which generates these numbers is $npp(p, k0, m, acc)$ where you first set series precision to be more than the greatest number you expect in the output, and then run with acc, the accuracy, more than this number (or more than three times it for $p = 2$); the other things are p, the prime, $k0$, the component, and m, the number of a_i you want. The code for *npp* is in *cps.g*.

3.6 $p = 59$

The point was that perhaps something funny was going on for $p = 59$. I'll let meccah node 5 work out the answer for me though.