Examples of Hida families.

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February 7, 2012

Last modified 17/05/2006.

1 The theory.

If $\mathbf{T}$ is a Hida Hecke algebra then it's finite and free over the local ring $\Lambda = \mathbf{Z}_p[[w]]$. One can then choose a Λ-basis and compute the discriminant of such a basis (e.g. as a determinant of a matrix coming from traces). We get a principal ideal of Λ which may tell us something about $\mathbf{T}$.

Note that multiplication by U_p is a Λ-endomorphism of $\mathbf{T}$ and hence by Cayley-Hamilton U_p satisfies a monic polynomial of some degree. On the other hand I don't know if $\Lambda[U_p]$ is free of the same rank as $\mathbf{T}$ in general.

Let's say $\mathbf{T}$ is free of rank 2 over Λ. Then we're suddenly in good shape. The reduction of $\mathbf{T}$ modulo the maximal ideal $\Updownarrow$ of Λ is a finite Hecke algebra of rank 2 over $\mathbf{F}_p$ and the element $1 \in \Lambda$ can be extended to a basis of this finite Hecke algebra, and a lift of the other element to $t \in \mathbf{T}$ gives us a Λ-basis $\{1, t\}$ for $\mathbf{T}$ over Λ. We have $t^2 + bt + c = 0$ and the discriminant of $\mathbf{T}$ is just $\Delta := b^2 - 4c$. This is an element of Λ so, if it's not zero, then it's of the form $up^\mu f$ with $f \in \mathbf{Z}_p[w]$ a distinguished polynomial (that is, it is monic and reduces mod p to a monomial) and u a unit in $\mathbf{Z}_p[[w]]$.

Frank Calegari and William Stein computed some examples of Δ using a Λ-adic calculation on overconvergent forms. Here is another trick which seems to work as well. Consider the subring $\Lambda[U_p]$ of $\mathbf{T}$. It's either Λ again, or $U_p = x + yt$ with $y \neq 0$ and we see that the sub*module* of $\mathbf{T}$ generated by 1 and U_p is $\Lambda \oplus yt\Lambda$. Now this is easily checked to be a sub*ring*, as $(yt)^2 = -cy^2 - by(yt)$, and it has a discriminant which one can easily check is $\Delta' := t^2\Delta$. Now discriminants are only defined up to a unit—but one can really compute the discriminant of the basis $\{1, U_p\}$ and get an actual element of Λ, whose specialisation to weight k is the actual discriminant of U_p on the weight k ordinary forms, not up to a unit, so we're getting a finer invariant here.

2 Examples of Hida Hecke algebras.

1) $\mathbf{T} = \Lambda$. For example $N = 1$ and $p = 11$ and the component of weight space corresponding to $k = 2$. There is one ordinary form of level 11 and so $\mathbf{T} = \Lambda$.

2) $\mathbf{T} = \Lambda \otimes_{\mathbf{Z}_p} W(\mathbf{F})$ for some finite extension $\mathbf{F}$ of $\mathbf{F}_p$. For an example with $\mathbf{F}$ quadratic take $N = 1$ and $p = 23$ and the component of weight space corresponding to $k = 2$. In this case the weight 2 specialisation of the Hecke algebra has discriminant 5 and is in fact the full integers of $\mathbf{Q}(\sqrt{5})$. Now 23 is inert in this field so somehow $\mathbf{T}$ is now free of rank 1 over $W(\mathbf{F}_{23^2})$ and somehow this is it. I don't know if there are any unanswered questions here. I am slightly confused by the residue field not being $\mathbf{F}_p$. If we base extend to $W(\mathbf{F}_{23^2})$ then the Hida family splits into two rank 1 things over $W(\mathbf{F}_{23^2})[[w]]$ because the ordinary forms are conjugate, not congruent.

3) Things in $\Lambda \oplus \Lambda$ which are congruent mod p. Frank is pretty sure such an example should exist, but doesn't remember if he knows an explicit one. Here was his strategy to find such an example: look for an elliptic curve $E/\mathbf{Q}$ of level prime to p such that $v_\ell(j) < 0$ furthermore $v_\ell(j)$ is congruent to 0 mod p, with ℓ a prime of multiplicative reduction for E of course. The point is that one can then level-lower and hope to get a newform and an oldform.

4) This example I understand pretty well: $\mathbf{T}$ free of rank 2 and with discriminant ideal (f) with $f = X - up^n$. In fact I know several examples of this.

Here's the first one, which Robert Pollack pointed me to. Frank and William also independently understood this example. Set $N = 11$ and $p = 3$ and look at the component coming from $k = 2$. One can compute the discriminant of T_3 on the ordinary weight k level N forms (of course one uses the forms of level prime to 3; the forms actually giving the Hida family are 3-old but the discriminant won't change much) for $k \geq 4$ an integer, and observe that for $k = 4$ (resp. $k = 6$) the discriminant of T_p on the ordinary forms is congruent to 3 (resp. 6) mod 9. This means that $\Delta' \in \mathbf{Z}_p[[w]]$ is $a + bw + O(w^2)$ and specialises to 3 (resp. 6) when $w = 4^4 - 1$ (resp. $4^6 - 1$). These values of w are congruent to 3 (resp. 0) mod 9 and this implies that b is a unit and that a isn't, so $\Delta' = uf$ with u a unit and $f = w - c$ with $3|c$. We don't really care what c is because w is chosen arbitrarily and so in fact we may as well set $c = 0$ (if $w = 4^k - 1$ then c is actually approximately 78). Note that it only took two computations of modular forms to see this.

We normalise so that the discriminant is w and deduce that $U_p^2 + BU_p + C = 0$ with $B^2 - 4C = uw$ and hence the discriminant ideal of $\mathbf{T}$ must be w because this is squarefree. Because $p \neq 2$ we can renormalise so that $B = 0$ and deduce that $\mathbf{T} = \Lambda[X]/(X^2 - w)$.

Here are other examples. If $k = 28$ and $N = 1$ and $p = 131$ then the same techniques show that $\mathbf{T} = \Lambda[X]/(X^2 - w)$ again. One computes the T_p-eigenvalues of the ordinary level 1 weight $28 + i(p - 1)$ forms for $i = 0, 1$ and the resulting discriminant. One divides the discriminant by p and computes the reduction mod p of the results: this is like computing $a + bw$ above. This should be a linear function in i. Of course doing it for $i = 0$ and $i = 1$ and observing that the reduction is non-constant is enough to give you $p|a$ and $p \nmid b$, if $\Delta' = a + bw + \ldots$, but if you do it for $i = 2, 3$ as well you get a check on what's going on: for $i = 0, 1, 2, 3$ the answers are $22, 103, 53, 3$ which are indeed in arithmetic progression mod 131.

One last example: $k = 28$ and $N = 1$ and $p = 139$. Same idea. Get $35, 44, 53$.

3 Questions.

In general $\mathrm{Disc}(\mathbf{T}) = up^{\mu} f$ with f distinguished. Barry asks if one can find examples with $\mu > 0$. Robert asks if one can find quadratic examples with $\deg(f) > 2$.