Dimension of spaces of Eisenstein series

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Abstract

For want of a reference, we give a formula for the dimension of the space of Eisenstein series in $M_k^{\text{new}}(\Gamma_1(N);\epsilon)$. On the way, we compute the dimension of the Eisenstein subspace of $M_k(\Gamma_1(N);\epsilon)$.

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The formula

Twice now, I have found myself working out a formula for the dimension of the space of Eisenstein series in $M_k(\Gamma_1(N);\epsilon)$, and/or the dimension of the subspsace of new Eisenstein series, and so I am following the example set by William Stein and writing a short LaTeX document explaining the computation this time, so I don't ever have to do it again. Notation: $N \geq 1$ is an integer, $k \geq 2$ is a weight (for weight 1 see the end of this note) and ϵ is a Dirichlet character of level N. The dimension of the space of Eisenstein series can be read off from the formula in [1], but we give a self-contained approach which deals with the new case too. The dimension of the space of Eisenstein series for $\Gamma_1(N)$ is well-known—for example, for $N \notin \{1,2,4\}$ the dimension is $\frac{1}{2}\sum_{0<d|N}\phi(d)\phi(N/d)$ for $k > 2$, where ϕ is the Euler ϕ-function, and one less than this for $k = 2$. Note that for $N \notin \{1,2,4\}$, the number $\frac{1}{2}\sum_{0<d|N}\phi(d)\phi(N/d)$ is the number of cusps on $X_1(N)$. The issue is what the characters of these Eisenstein series are. The way I've dealt with this is simply to look at the proof I know of the above formula, which goes "here is a construction of this many Eisenstein series, and this is a basis for the space", and to compute the characters of the Eisenstein series constructed.

Let's let $e_k(N;\epsilon)$ denote the dimension of the space of Eisenstein series of weight $k \geq 2$, level N and character ϵ, and let's let $e_k^{\text{new}}(N;\epsilon)$ denote the dimension of the new subspace of this space. Of course, if $\epsilon(-1) \neq (-1)^k$ then the answer is trivially $e_k(N;\epsilon) = e_k^{\text{new}}(N;\epsilon) = 0$ so let's also assume that $\epsilon(-1) = (-1)^k$ from now on. For $N \leq 4$ my gut feeling is to do things by hand—this is easy because there is at exactly one even and at most one odd character, so one can read things off from formulae for the $\Gamma_1(N)$ case. We summarise the answers below:

If $N=1$ then $\epsilon=1$ and $e_k(N;\epsilon)=1$ (a newform, of course) for $k\geq 4$ even and 0 otherwise.

If $N=2$ then again $\epsilon=1$ and so we can assume k is even and then $e_k(N;\epsilon)=2$ for k even with $k\geq 4$, both forms being oldforms, and $e_2(N;\epsilon)=1$, this being a newform, even though it has all the appearance of an oldform.

If $N=3$ then either $\epsilon=1$ in which case k should be even for the answer to be non-zero, and in this case $e_k(N;\epsilon)=2$ for $k>2$ (both old) and $e_2(N;\epsilon)=1$ (new), or ϵ is non-trivial in which case k should be odd and then $e_k(N;\epsilon)=2$ (both new).

If $N=4$ then again either $\epsilon=1$ in which case k should be even and $e_k(N;\epsilon)=3$ (all old) for $k\geq 4$ and $e_2(N;\epsilon)=2$ (both old), or ϵ is non-trivial, in which case $k\geq 3$ is odd and $e_k(N;\epsilon)=2$ (both new). Note that this latter answer differs from the number of cusps on $X_1(4)$, which is 3, because of problems at the "middle cusp", where the stack $X_1(4)$ differs from the scheme $X_1(4)$, or however one wants to look at it, and this forces any modular form of odd weight to vanish at this cusp.

So let's assume now that $N\geq 5$ (by the way, I am not 100% sure that this restriction is necessary, it's just an assumption that makes me more at ease with the theory), $\epsilon:(\mathbf{Z}/N\mathbf{Z})^\times\to\mathbf{C}^\times$, $k\geq 2$, and $(-1)^k=\epsilon(-1)$. In this case, the answer does not depend on k if k is odd, and if k is even then $e_k(N;\epsilon)$ is independent of $k\geq 2$ if ϵ is non-trivial, but for $\epsilon=1$ we have $1+e_2(N;\epsilon)=e_k(N;\epsilon)$ for any $k\geq 4$. Hence we may as well assume that $k\geq 3$ and $(-1)^k=\epsilon(-1)$. Now, by the constructive proof of the well-known formula for the dimension of the space of Eisenstein series, $e_k(N;\chi)$ equals the size of the set

$$\{(d,\chi,\psi):0<d|N,\chi:(\mathbf{Z}/d\mathbf{Z})^\times\to\mathbf{C}^\times,\psi:(\mathbf{Z}/(N/d)\mathbf{Z})^\times\to\mathbf{C}^\times,\chi\psi=\epsilon\},$$

as for each (d,χ,ψ) one can construct an explicit Eisenstein series and these forms are a basis for the space we're trying to compute the dimension of. The l-adic Galois representation associated to the form is $\tilde{\chi}\oplus\tilde{\psi}\omega^{k-1}$, where ω is the cyclotomic character and a tilde over a Dirichlet character denotes the associated (finite image) Galois representation. Furthermore, such a form is new iff $\mathrm{Cond}(\chi)=d$ and $\mathrm{Cond}(\psi)=N/d$. So now the problem is reduced to combinatorics.

One pleasant consequence of this explicit formula above is that it is a product of local factors, in the sense that if $\epsilon=\prod_p\epsilon_p$ is the expression of ϵ as a product of factors of prime-power level, and $N=\prod_p p^{e(p)}$, then $e_k(N;\epsilon)=\prod_p e_k(p^{e(p)};\epsilon_p)$ and similarly for the newforms. So this reduces our computation to the case where N is a prime power.

Now we have to get our hands dirty. Let's say $N=p^r$ and ϵ has conductor p^s with $s\leq r$. Of course we assume $r\geq 1$. For d fixed, ψ and χ determine each other, and sometimes one is easier to count than the other.

First we count the space of all Eisenstein series. Let's first assume that $2s\leq r$. We sum over d first. If $d\leq p^{r/2}$ then any χ will work, and these contribute $1+\phi(p)+\phi(p^2)+\ldots+\phi(p^{\lfloor r/2\rfloor})$. For higher d we count ψ instead.

The total sum depends on whether r is odd or even, because of the middle term. If $r = 2r'$ then the total is $1 + (p-1) + p(p-1) + \ldots + p^{r'-1}(p-1) + p^{r'-2}(p-1) + \ldots + 1 = p^{r'} + p^{r'-1}$. But if $r = 2r'+1$ then we add another $p^{r'-1}(p-1)$ to the above sum and get $2p^{r'}$.

Now let's consider the case $r < 2s$. In this case, the conductor of either ϕ or χ has to be at least p^s and this means that precisely one of the two characters has conductor at least p^s. By symmetry we can then just compute the pairs (χ, ψ) for $d = p^t$ with $0 \leq t \leq (r-s)$ and then double the answer. One counts the possibilities for χ and gets $1+\phi(p)+\phi(p^2)+\ldots+\phi(p^{r-s}) = \sum_{d|p^{r-s}} \phi(d) = p^{r-s}$ and hence the total is $2p^{r-s}$.

What we have proved is that if $N = p^r$ and ϵ has conductor p^s then the dimension of the space of Eisenstein series of level N and character ϵ is $\lambda(r, s, p)$, with notation as in [1]. Hence the dimension of the space of Eisenstein series, in the general case, under our assumptions $N \geq 5$, $k \geq 3$ and $(-1)^k = \epsilon(-1)$, is $\prod_{p|N} \lambda(r_p, s_p, p)$, where $p^{r_p}||N$ and $p^{s_p}||\operatorname{Cond}(\epsilon)$. This formula can also easily be deduced from Theorem 1 of [1], but perhaps our proof is more direct.

Now let's do the newforms. Again we work locally, so $N = p^r$ and ϵ has conductor p^s as before. Certainly if $\phi\psi = \epsilon$ then the conductor of either ϕ or ψ is at least p^s. Hence if $2s > r$ then because we are in the new case, the conductor of precisely one of these characters is p^s and the conductor of the other is p^{r-s}. So $e_k^{\text{new}}(N;\epsilon)$ is simply twice the number of characters of conductor p^{r-s}. If $r = s$ then the answer is hence 2, if $r - s = 1$ then the answer is $2p-4$, and if $r-s > 1$ then the answer is $2\phi(p^{r-s}) - 2\phi(p^{r-s-1}) = 2(p-1)^2p^{r-s-2}$.

If $2s = r$ then we have to count the number of characters χ of conductor p^s such that ϵ/χ also has conductor p^s. If $p = 2$ then this is 0, because $1 + 2^{s-1}$ must be sent to -1 by both χ and ϵ. If $p > 2$ then we just count: the case $s = 1$ gives us $(p-3)$ possibilities, and for $s > 1$ we must make sure that $1 \neq \chi(1+p^{s-1}) \neq \epsilon(1+p^{s-1})$ which gives us $\frac{p-2}{p}\phi(p^s) = (p-2)(p-1)p^{s-1}$ as the solution.

If finally $2s < r$ then for r odd there are no newforms, because the newness assumption $\operatorname{Cond}(\chi)\operatorname{Cond}(\psi) = p^r$ implies that the conductors of χ and ψ are distinct, hence the max of these two values will be p^s which is too small. In the case $r = 2r'$, the same analysis shows us that χ and ψ must both have conductor $p^{r'}$. One character is determined by the other, which can be arbitrary of conductor $p^{r'}$ so again the solution is given by counting the number of characters of conductor $p^{r'}$, which is $\phi(p^{r'}) - \phi(p^{r'-1}) = p-2$ if $r = 2$ and $(p-1)^2(p^{r'-2})$ if $r > 2$. I'm doing these off the top of my head now so I could be making mistakes—I have never checked any of these formulae.

The general solution in the new case can be built up from these building blocks, as in the other case.

Appendix: weight 1. In the weight 1 case the formula in [1] gives the dimension of the Eisenstein series when you specialise to weight 1; it gives you the difference between the modular form and the cusp form dimensions. It simplifies to

$$\frac{1}{2}\prod_{p|N}\lambda(r_p, s_p, p)$$

(with notation as above) (and so only depends on N and the conductor of ϵ) (here we are assuming that $\epsilon(-1) = -1$ of course). The 1/2 is there because, basically, the Eisenstein series associated to (χ, ψ) is the same as the one associated to (ψ, χ) and because $\chi\psi(-1) = -1$ one can never have $\chi = \psi$. Miyake is a useful reference for explicit formulae in this case. If χ has conductor L and ψ has conductor M and $\chi\psi(-1) = -1$ and χ is even then there is a newform of level LM and its q-expansion is $C + \sum_{n\geq 1}(\sum_{0<d|n}\chi(n/d)\psi(d))q^n$ where $C = 0$ if $L > 1$ and $C = L(0, \psi)/2$ if $L = 1$. This will now give everything. Note finally for reference that if ψ is odd then $L(0, \psi) = -\frac{1}{M}\sum_{a=1}^{M} a\psi(a)$.

References

[1] H. Cohen and J. Oesterlé, *Dimensions des epaces de formes modulaires*, Springer Lecture Notes in Maths 627, pp69–78.