Notes on the definitions of group cohomology and homology.

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VERY sloppy notes on homology and cohomology. Needs work in several places. Last updated 3/12/07.

1 Derived functors.

My goal is to define Ext and Tor but there's no harm talking about general nonsense for a bit. Because I want to derive both Hom and $\otimes$ with respect to both variables, we're going to have to talk about functors which are either left exact or right exact, and functors which are either covariant or contravariant. But there's a trick. If A is an abelian category then A^{op} is too, and so we can work out the theory just with right derived functors of a covariant left exact functor $F : A \to B$, and then apply the theory with functors $A \to B^{op}$, $A^{op} \to B$ and $A^{op} \to B^{op}$!

So first let's say A and B are abelian categories, and F is a covariant, left exact functor (that is, if $0 \to L \to M \to N \to 0$ is exact then so is $0 \to F(L) \to F(M) \to F(N)$).

If A has enough injectives then left exact (covariant) functors have right derived functors, and they can be computed in the usual way: if X is an object of A then write down an injective resolution $0 \to X \to I^0 \to I^1 \to \cdots$, apply the functor and get a complex $0 \to F(I^0) \to F(I^1) \to \cdots$, and define the right derived functor $R^n(F)$ to be the cohomology of this complex at the point $F(I^n)$. Note for example that $R^0(F) = F$ by left exactness. We recall the basic property of the derived functor in this case: if $0 \to X \to Y \to Z \to 0$ is exact, then so is

$$0 \to F(X) \to F(Y) \to F(Z) \to R^1F(X) \to R^1F(Y) \to \cdots .$$

Next, if F is a *contravariant* functor $A \to B$ then think of F as a covariant functor $\tilde{F} : A^{op} \to B$; it seems that the standard terminology is that F is *left exact* if, by definition, $\tilde{F}$ is left exact.

If F is a contravariant functor which is left exact in this sense, then we will be able to pull off the same trick if A^{op} has enough injectives, which of course is the same as A having enough projectives. The functor $\tilde{F}$ of course has right derived functors $R^n\tilde{F} : A^{op} \to B$ which can be thought of as contravariant functors $R^nF : A \to B$—these are again called the right derived functors of the contravariant left exact functor F. Explicitly, if $X \in A$ and $\cdots \to P_1 \to P_0 \to X \to 0$ is a projective resolution, and if we apply F, then we get $0 \to F(P_0) \to F(P_1) \to \cdots$ and we take the cohomology of this at the $F(P_n)$ term to get $R^nF(X)$. And the canonical exact sequence is that if $0 \to X \to Y \to Z \to 0$ is exact then so is $0 \to F(Z) \to F(Y) \to F(X) \to R^1F(Z) \to R^1F(Y) \to \cdots$. As a sanity check: we computed using projective resolutions so if Z is projective then $R^1F(Z) = 0$, but this is unsurprising because in this case the short exact sequence splits.

A right exact covariant functor $A \to B$ can be thought of as a left exact covariant functor $A^{op} \to B^{op}$, so the left derived functors of a right exact functor can be computed using projective resolutions in A, if A has enough projectives. Similarly a right exact contravariant functor $A \to B$, that is, a contravariant functor for which the associated covariant functor $A^{op} \to B$ is right exact, will have left derived functors iff the associated map $A \to B^{op}$ has right derived functors, so if A has enough injectives then the derived functors exist and you can compute them using injective resolutions in A.

2 Example of a projective resolution.

This will come in handy later. Let G be a group, let A be the ring $\mathbf{Z}[G]$, and consider the category of left A-modules (that is, A acts on the left). Here is a "standard" projective resolution of $\mathbf{Z}$, considered as a left G-module with G acting trivially. It's called the "unnormalised bar resolution".

Let B_n (for $n \geq 0$) be the free A-module on the set of all symbols $(g_1, g_2, \ldots, g_n)$. Thought of another way, B_n is the free $\mathbf{Z}$-module on the set of all symbols $[g_0, g_1, \ldots, g_n] := g_0(g_1, g_2, \ldots, g_n)$. For $n \geq 1$ define an A-linear map $d : B_n \to B_{n-1}$ by demanding that on A-generators it does this:

$$d(g_1, \ldots, g_n)$$

$$= g_1(g_2, \ldots, g_n) - (g_1g_2, g_3, \ldots, g_n) + (g_1, g_2g_3, g_4, \ldots, g_n) - \cdots$$
$$\cdots + (-1)^{n-1}(g_1, g_2, \ldots, g_{n-1}g_n) + (-1)^n(g_1, g_2, \ldots, g_{n-1}).$$

So, explicitly, $d : B_1 \to B_0$ sends the A-generator (g) to $g(*) - ()$ with $()$ the generator of B_0 (which is free of rank 1) and $d : B_2 \to B_1$ sends (g, h) to $g(h) - (gh) + (g)$.

Finally define $\epsilon : B_0 \to \mathbf{Z}$ by sending $g(*)$, for $g \in G$, to 1. The claim, which I won't prove, is that $\cdots \to B_2 \to B_1 \to B_0 \to \mathbf{Z} \to 0$ is exact. I suspect that one relatively painless proof of this involves using some other basis where d has a much simpler definition! Another proof is hinted at in Exercise 6.5.1(1) of Weibel. But let's press on, I'll prove enough to be able to compute H_1 and H^1 group homology and cohomology.

Checking that the sequence is exact at the $\mathbf{Z}$ and B_0 terms can be done by hand: clearly B_0 surjects onto $\mathbf{Z}$, and if one thinks of B_0 as the ring A with $(*) = 1$, then the map $B_1 \to B_0$ sends (g) to $g - 1$, and such things generate (as a left ideal) the kernel of $B_0 \to \mathbf{Z}$, as is easily checked. Let me check exactness at the B_1 term, by mimicing the general proof. To check exactness here I firstly have to check that $d^2 = 0$, and it suffices to check this on generators. If $(g, h) \in B_2$ then $d(g, h) = g(h) - (gh) + (g)$ and d of this is $gh(*) - g(*) - gh(*) + (*) + g(*) - (*) = 0$, so $d^2 = 0$, and to check it exactness I just need to check that anything in the kernel of $d : B_1 \to B_0$ is in the image of $d : B_2 \to B_1$. Here's how I'll do this: define $s : B_1 \to B_2$, $\mathbf{Z}$-linear but not A-linear, by $s(g(h)) = (g, h)$, and define $s : B_0 \to B_1$, $\mathbf{Z}$-linear but not A-linear, by $s(g(*)) = (g)$. I claim that $ds + sd$ is the identity map $B_1 \to B_1$, and this will prove exactness at B_1. It suffices to check on $\mathbf{Z}$-generators, so it suffices to check that $(ds + sd)(g(h)) = g(h)$. But this is easy: $(ds)(g(h)) = d(g, h) = g(h) - (gh) + (g)$, and $(sd)(g(h)) = s(gd(h)) = s(gh(*) - g(*)) = (gh) - (g)$ so $ds + sd$ is the identity. But this is enough to prove that the kernel of $d : B_1 \to B_0$ coincides with the image of $d : B_2 \to B_1$!

We can also make the B_n free right A-modules by switching the G-action in the usual way: $b * g := g^{-1}b$. Using this trick we also get a projective resolution of $\mathbf{Z}$ in the category of right A-modules.

3 Ext.

Let A be a ring, not necessarily commutative. Write A-Mod for the category of left A-modules (that is, A acts on the left) and Mod-A for the category of right A-modules. Both of these categories have enough injectives and enough projectives.

If N is a left A-module then $\mathrm{Hom}_A(N, *)$ is covariant and left exact, so it has right derived functors which are called $\mathrm{Ext}^n_A(N, *)$, and $\mathrm{Ext}^n_A(N, X)$ can be computed using an injective resolution of X. We deduce immediately that if $0 \to X \to Y \to Z \to 0$ is a short exact sequence of left A-modules then we have a long exact sequence

$$0 \to \mathrm{Hom}_A(N, X) \to \mathrm{Hom}_A(N, Y) \to \mathrm{Hom}_A(N, Z) \to \mathrm{Ext}^1_A(N, X) \to \mathrm{Ext}^1_A(N, Y) \to \cdots.$$

As a sanity check: if X is injective then $0 \to X \to Y \to Z \to 0$ splits so it looks like $\mathrm{Ext}^1_A(N, X)$ should be zero, and it is because X is an injective resolution of X and this complex has no term in degree 1.

Now consider a left A-module X and the functor $\mathrm{Hom}_A(*,X)$. This functor is contravariant and if $M \to N$ is surjective then $\mathrm{Hom}_A(N,X) \to \mathrm{Hom}_A(M,X)$ is injective, so $\mathrm{Hom}_A(*,X)$ is left exact and contravariant, so it has right derived functors, and these functors can be computed using a projective resolution of N (because it's injective in the opposite category to A-Mod).

The funny thing is that these derived functors are just $\mathrm{Ext}^n_A(*,X)$ again! The reason for this is that if N and X are left R-modules and I^* is an injective resolution of X and P_* is a projective resolution of N, then there is a double complex called $\mathrm{Hom}(P_*,I^*)$ and the associated total complex has natural maps to $\mathrm{Hom}_(P_*,X)$ and to $\mathrm{Hom}_A(N,I^*)$ and both of these natural maps are quasi-isomorphisms. This is explained much more carefully in section 2.7 of Weibel's homological algebra book.

We deduce that if $0 \to L \to M \to N \to 0$ is a short exact sequence of left A-modules then we get a long exact sequence

$$0 \to \mathrm{Hom}_A(N,X) \to \mathrm{Hom}_A(M,X) \to \mathrm{Hom}_A(L,X) \to \mathrm{Ext}^1_A(N,X) \to \mathrm{Ext}^1_A(M,X) \to \cdots .$$

The sanity check: if N is projective then the sequence splits, so $\mathrm{Ext}^1_A(N,X)$ should be zero, and it is because we computed it using a projective resolution of N.

Ext is not symmetric (as Hom isn't!). But it is additive in each variable and contravariant in the first, covariant in the second (so, for example, an A-module hom $N_1 \to N_2$ induces maps $\mathrm{Ext}^n_A(N_2,L) \to \mathrm{Ext}^n_A(N_1,L)$, "because it's true for $n=0$").

4 Group cohomology.

If G is a group and M is a G-module then $H^i(G,M)$ is just $\mathrm{Ext}^i_A(\mathbf{Z},M)$, where $A=\mathbf{Z}[G]$. Here $\mathbf{Z}$ is the A-module defined by $g.z=z$ for all $g \in G$ and $z \in \mathbf{Z}$. This can be computed by either using an injective resolution of M, which is typically going to be hell to write down, or a projective resolution of $\mathbf{Z}$, and we wrote one of them down earlier! We deduce that $H^i(G,M)$ is the homology of the complex whose ith term is $\mathrm{Hom}(B_i,M)$. We have

$$0 \to \mathrm{Hom}_A(B_0,M) \to \mathrm{Hom}_A(B_1,M) \to \mathrm{Hom}_A(B_2,M) \to \cdots .$$

Because the B_i are free A-modules of rank "G^i", as it were, we may consider $\mathrm{Hom}_A(B_i,M)$ as just the (set-theoretic) maps $G^i \to M$. So we see

$$0 \to M \to \mathrm{Hom}(G,M) \to \mathrm{Hom}(G^2,M) \to \cdots .$$

The elements of $C^i := \mathrm{Hom}(G^i,M)$ are called i-cochains, and an i-cochain in the kernel of d is an i-cocycle, and one in the image of d is an i-coboundary. The cohomology of M is the homology of this complex. We see explicitly that dm is the map $G \to M$ sending g to, well, you do dg in B_0, and get $g(*)-(*)$, and then consider the A-linear map sending $(*)$ to M and you get $gm-m$, so there's d from C^0 to C^1; its kernel is M^G so we recover that $H^0(G,M)=M^G$. Similarly $d: C^1 \to C^2$ is defined thus: if $f: G \to M$ then df is the map $G^2 \to M$ such that $(df)(g,h)=gf(h)-f(gh)+f(g)$, so a 1-cocycle is a function f with $f(gh)=gf(h)+f(g)$ and a 1-coboundary is f of the form $f(g)=gm-m$. In particular if G acts trivially on M then a 1-cocycle is just a group homomorphism and all coboundaries are zero, so $H^1(G,M)$ is just the group homomorphisms $G \to M$.

5 Tor.

If N is a right A-module then $N \otimes_A *$ is a covariant right exact functor from left A-modules to abelian groups (or even to A-modules, if A is commutative), so this functor has left derived functors, and they're called $\mathrm{Tor}^A_n(N,*)$ and for X a left A-module we can compute $\mathrm{Tor}^A_n(N,X)$

using a projective resolution of X. Given a short exact sequence $0 \to X \to Y \to Z \to 0$ of left A-modules we have a long exact sequence

$$\cdots \to \mathrm{Tor}_1^A(N,Y) \to \mathrm{Tor}_1^A(N,Z) \to N \otimes_A X \to N \otimes_A Y \to N \otimes_A Z \to 0$$

and the sanity check is that if Z is projective then $\mathrm{Tor}_A^1(N,Z)$ will vanish and the exact sequence will split.

Now if X is a left A-module then $* \otimes_A X$ is right exact and covariant on the category of right A-modules, so we can take left derived functors, and, although I won't give them a name yet, they can be computed using a projective resolution of N in the usual way. Now the funny thing is that in fact these derived functors are called $\mathrm{Tor}_n^A(*,X)$ again, because they're isomorphic to the things above! The reason for this is the following. If N is a right A-module and X is a left A-module and if P_* is a projective resolution of N and Q_* is a projective resolution of X, then the complex $P_* \otimes_A X$ is quasi-isomorphic with the total complex of the double complex $(P_i \otimes_A Q_j)_{i,j}$, which in turn is quasi-isomorphic to the complex $N \otimes_A Q_*$. The natural maps are from the total complex to the "smaller" complexes, so in particular it's perhaps going to be tricky to spot a natural map from one definition of $\mathrm{Tor}_n^A(N,X)$ to the other!

The upshot, of course, is that if $0 \to L \to M \to N \to 0$ then we have a long exact sequence

$$\cdots \to \mathrm{Tor}_1^A(M,X) \to \mathrm{Tor}_1^A(N,X) \to L \otimes_A X \to M \otimes_A X \to N \otimes_A N \to 0$$

and again the sanity check is that if N is projective then $\mathrm{Tor}_1^A(N,X) = 0$ and the sequence splits.

Note that flat modules are acyclic for tensor product functors so their higher Tor groups vanish, and one can use flat resolutions to compute Tor groups.

6 The Künneth formula.

Let $\cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots$ be a chain complex of torsion-free abelian groups and let M be an abelian group. How does the homology of $P_* \otimes M$ relate to M tensored with the homology of P_*? Because of our torsion-freeness assumption, it's not too hard to see a relation.

Theorem 1. *For every n there's an exact sequence*

$$0 \to H_n(P) \otimes_{\mathbf{Z}} M \to H_n(P \otimes_{\mathbf{Z}} M) \to \mathrm{Tor}_1^{\mathbf{Z}}(H_{n-1}(P), M) \to 0.$$

Proof. Let Z_n be the kernel of $P_n \to P_{n-1}$ and let B_{n-1} be the image of $P_n \to P_{n-1}$. Then $0 \to Z_n \to P_n \to B_{n-1} \to 0$, and since P_{n-1} is torsion-free we deduce that B_{n-1} is too, so B_{n-1} is flat as a $\mathbf{Z}$-module and so $\mathrm{Tor}_1^{\mathbf{Z}}(B_{n-1}, M) = 0$, so

$$0 \to Z_n \otimes M \to P_n \otimes M \to B_{n-1} \otimes M \to 0$$

is exact. Put these together to get a short exact sequence of chain complexes

$$0 \to Z_* \otimes M \to P_* \otimes M \to B_{*-1} \otimes M \to 0$$

and consider what the associated long exact sequence looks like. The differentials in the Z and B complexes are zero! So the long exact sequence looks like this:

$$\cdots \to H_{n+1}(B_{*-1} \otimes M) \to H_n(Z_* \otimes M) \to H_n(P_* \otimes M) \to H_n(B_{*-1} \otimes M) \to H_{n-1}(Z_* \otimes M) \to \cdots$$

and when translated into English we get

$$\cdots \to B_n \otimes M \to Z_n \otimes M \to H_n(P_* \otimes M) \to B_{n-1} \otimes M \to Z_{n-1} \otimes M \to \cdots.$$

Now $0 \to B_n \to Z_n \to H_n(P_*) \to 0$, and because tensor is right exact we deduce

$$0 \to H_n(P) \otimes M \to H_n(P_* \otimes M) \to \ker(B_{n-1} \otimes M \to Z_{n-1} \otimes M) \to 0.$$

Finally $0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(P_*) \to 0$ is a flat resolution of $H_{n-1}(P_*)$ in the category of abelian groups, so the kernel above is $\mathrm{Tor}_1^{\mathbf{Z}}(H_{n-1}(P_*), M)$. $\square$

7 Group homology.

If G is a group and M is a left G-module then $H_0(G, M)$ is defined to be the biggest quotient of M where G acts trivially. Of course in this situation M is a left A-module, if $A = \mathbf{Z}[G]$, and we easily check that $H_0(G, M) = \mathbf{Z} \otimes_A M$, where $\mathbf{Z}$ is the right A-module with trivial G-action. We define $H_n(G, M) = \mathrm{Tor}_n^A(\mathbf{Z}, M)$, so if $0 \to L \to M \to N \to 0$ is a short exact sequence of G-modules (with G acting on the left) then there's an associated long exact sequence

$$\cdots \to H_1(G, M) \to H_1(G, N) \to H_0(G, L) \to H_0(G, M) \to H_0(G, N) \to 0$$

and furthermore $H_n(G, M)$ can be computed using a projective resolution of either $\mathbf{Z}$ or M. But we already have a projective resolution of $\mathbf{Z}$! Before we apply it let's see what comes from Künneth though: we get facts about when G acts trivially on M.

If G acts trivially on M then $H_0(G, M) = M$. If I is the augmentation ideal in $\mathbf{Z}[G]$ then

$$0 \to I \to \mathbf{Z}[G] \to \mathbf{Z} \to 0$$

is a short exact sequence of right $\mathbf{Z}[G]$-modules, the middle one of which is projective, and hence if $A = \mathbf{Z}[G]$ then for any module M we have

$$0 \to \mathrm{Tor}_1^A(\mathbf{Z}, M) \to I \otimes_A M \to M \to \mathbf{Z} \otimes_A M \to 0.$$

If $M = \mathbf{Z}$ then we deduce $0 \to H_1(G, \mathbf{Z}) \to I/I^2 \to \mathbf{Z} \to \mathbf{Z} \to 0$ and the map $\mathbf{Z} \to \mathbf{Z}$ is an isomorphism, hence $H_1(G, \mathbf{Z}) = I/I^2 = G^{ab}$, the map being the one sending $g - 1 \in I$ to g (this works).

Now let's apply the universal coefficient theorem. Let M be any abelian group with trivial G-action. Write down any projective resolution Q_* of $\mathbf{Z}$ by free right A-modules, and let $P_* = Q_* \otimes_A \mathbf{Z}$. Then the Q_n were free A-modules, so the P_n are free $\mathbf{Z}$-modules, the homology of P_* is $H_n(G, \mathbf{Z})$ and the homology of $P_* \otimes_{\mathbf{Z}} M = Q_* \otimes_A M$ is $H_n(G, M)$. We know that $H_0(G, M) = M$ but the universal coefficient theorem lets us deduce that for all $n \geq 1$ we have a short exact sequence

$$0 \to H_n(G, \mathbf{Z}) \otimes M \to H_n(G, M) \to \mathrm{Tor}_1^{\mathbf{Z}}(H_{n-1}(G, \mathbf{Z}), M) \to 0,$$

and in particular setting $n = 1$ gives us $G^{ab} \otimes M = H_1(G, M)$ (the Tor_1 vanishes because $\mathbf{Z}$ is flat as a $\mathbf{Z}$-module).

Now let's try and do the general calculation of $H_1(G, M)$, when M is not assumed to have the trivial action. We consider the unnormalised bar resolution B_* of $\mathbf{Z}$, rigged so that A is acting on the right this time, and we tensor this over A with M and take homology. We see that we need to compute the homology of

$$B_2 \otimes_A M \to B_1 \otimes_A M \to B_0 \otimes_A M \to 0.$$

Now $B_0 = A$, B_1 is a free A-module on the set G, and B_2 is free on the set $G \times G$. So we can think of $C_2 := B_2 \otimes_A M$ as $\oplus_{g,h \in G}(g, h)M$, $C_1 := B_1 \otimes_A M$ as $\oplus_{g \in G}(g)M$ and $C_0 := B_0 \otimes_A M$ as M. The C_i are called i-cochains. The map $C_1 \to C_0$ sends $(g)m$ to $g^{-1}m - m$ (the inverse is there because we're thinking of B_1 as a right A-module) and the map $M_2 \to M_1$ sends $(g, h)m$ to $(h)g^{-1}m - (gh)m + (g)m$. One checks readily that $d^2 = 0$, that the cokernel of $M_1 \to M_0$ is the quotient of M by the group generated by $\{g^{-1}m - m\}$ ($g \in G$ and $m \in M$), and one readily checks that this is G-stable, so we recover the fact that $H_0(G, M) = M_G$.

The new thing that we see is that $H_1(G, M)$ is the quotient of the "1-cycles" by the "1-boundaries". By this I mean of course the kernel over the image. Now $\oplus_{g \in G} gM$ can be thought of as maps $f : G \to M$ which vanish away from a finite set. Such a thing is a 1-cycle iff $\sum_{g \in G}(g^{-1}f(g) - f(g)) = 0$, and a 1-boundary iff there's $j : G \times G \to M$ with finite support such that, well, it's a complete mess isn't it. It's basically $f(g) = \sum_{h \in G} h^{-1}j(h, g) - \sum_{h \in G} j(h, h^{-1}g) + \sum_{h \in G} j(g, h)$. Again one can check that this is a 1-cycle. I am embarassed to say that I did this.

Note in particular that in the abelian case, any map $G \to M$ with finite support is a 1-cycle, and the 1-boundaries are generated by the image of the maps $G \times G \to M$ which are zero away from one element $(g,h) \in G^2$, where they take the value m. Now d of such a thing is $(h)m + (g)m - (gh)m$, so when we take the quotient we recover an element of $G^{ab} \otimes M$, by explicit definition of the tensor product!

8 Restriction and corestriction.

If $H \subseteq G$ is a subgroup, and A is a left G-module, then $H_*(H, A)$ is a δ-functor on left G-modules, so there's a map to the universal δ-functor; this is just a fancy way of saying that there's a natural map $H_*(H, A) \to H_*(G, A)$. This map is called corestriction. Just the same argument shows that there is a natural map $H^n(G, A) \to H^n(H, A)$, and this map is called restriction.

If furthermore H is normal, then for a G/H-module M homology $H_*(G, M)$ is a δ-functor on G/H-modules, so there's a map $H_n(G, M) \to H_n(G/H, M)$ and this is called co-inflation; if N is a G-module then there's a natural surjection of G-modules $N \to N_H$ and hence a natural map $H_n(G, N) \to H_n(G, N_H) \to H_n(G/H, N_H)$ and this is also called co-inflation.

Similarly (H still normal in G) there's an inflation map $H^n(G/H, N^H) \to H^n(G, N^H) \to H^n(G, N)$.

Finally, if H is a normal subgroup *of finite index* in G, [1] and A is a G-module then there's 1
a "trace" map $A_G \to A_H$, defined as follows[2]. Write $G = \coprod_{i=1}^n Hg_i$ and for $a \in A$ define 2
$Ta = \sum_i g_i a$. This is certainly a map of abelian groups $A \to A$ and so it induces a map $A \to A_H$. Now if $a \in A$ and $g \in G$ then $T(ga - a) = \sum_i g_i ga - g_i a$ and because $g_i g = h_i g_{\sigma(i)}$ for $h_i \in H$ and σ a permutation of $\{1, 2, \ldots, n\}$ (g acting on the right on the cosets) we see that $T(ga - a) = \sum_i (h_i g_{\sigma i} a - g_{\sigma(i)} a)$ is zero in A_H. Hence we get an induced map $A_G \to A_H$. By universality we get induced trace maps $H_n(G, A) \to H_n(H, A)$, and the lemma is that if you first do trace and then corestriction you get multiplication by the index (so this result is about an endomorphism of the homology of the bigger group). This won't be true for endomorphisms of the homology of the smaller group, that is, if you do corestriction and then trace, I don't think you get multiplication by anything, even if H is normal in G: it's not even true for H_0. For example think about G a finite group, $H = 1$ and M a G-module. The map $M_G \to M_G$ induced by sending m to $\sum_g gm$ is multiplication by the order of G, but the map $M \to M$ (and note $M = H_0(H, M)$ induced by this is some kind of trace map still.

Similarly there's a trace $A^H \to A^G$ and hence maps $H^n(H, A) \to H^n(G, A)$ also called trace, and if you do restriction and then trace you get multiplication by the index.

Here's what the trace looks like on H_1. Let G and H be as above, and for $g \in G$ write $g_i g = h_i g_{\sigma(i)}$ as above. If $(g)m$ is a 1-chain for G, define its trace to be $\sum_i (g_i g g_{\sigma(i)}^{-1}) g_i m$. One checks easily that this is right, because it commutes with d. Note also that if G acts freely on $M = \mathbf{Z}$ then we're recovering the transfer map.

9 Some explicit recipes for connecting homs.

If $0 \to A \to B \to C \to 0$ is exact with a left G-action then the induced map $H_1(G, C) \to H_0(G, A)$ is defined thus: take a 1-cycle $\sum_{i=1}^n (g_i) c_i$ representing the homology class, and lift each c_i to b_i; we get $\sum_{i=1}^n (g_i) b_i$, a 1-chain in B. This isn't a 1-cycle necessarily though, because d of it is $\sum_i (g_i^{-1} b_i - b_i)$, an element of B that maps to zero in C. Hence it's in A. This is the boundary map, as one can check using the bar resolution, basically.

If $K \triangleleft G$ is a normal subgroup with $G/K = \Gamma$ and if M is a G-module then G acts natually on M_K (easy check: $g(km - m) = k'm' - m'$ with $k' = gkg^{-1} \in K$ and $m' = gm$), and hence Γ does too. In fact Γ will act on $H_n(K, M)$. Here's the explicit definition on H_1: in fact let's consider the obvious G-action on the 1-chains, where g sends $(k)m$ to $(gkg^{-1})gm$. One checks easily that this definition commutes with the boundary map above, so it's the right one.

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[1] **DO WE NEED NORMAL? THIS IS AN IMPORTANT QUESTION. MY SOURCE WAS WEIBEL! WEIBEL SAYS NORMAL! WHY??**

[2] This can't be the right name! It looks nothing like a trace!

10 Spectral sequences.

There's one for homology! The exact sequence of low degree terms looks like this. The set-up: K a normal subgroup of a group G, and A a left G-module. We have

$$H_2(G, A) \to H_2(G/H, A_H) \to H_1(H, A)_{G/H} \to H_1(G, A) \to H_1(G/H, A_H) \to 0.$$

Let me make explicit the Galois action on $H_1(H, A)$. In fact the Galois action lifts to the chains: g sends $(h)a$ to $(ghg^{-1})(ga)$. Again one can check that this is right because it commutes with d.