Computing $H^1(SL(2, k), \text{End}^0(k^2))$ on a computer.

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Here’s how I computed some group cohomology for $\text{End}^0(k^2)$ in MAGMA. First I wrote a program by hand which outputted how a $2 \times 2$ matrix in $SL(2, k)$ acted on the trace zero endomorphisms on $k^2$:

```magma
end0:=function(g)
    alp:=g[1,1]; bet:=g[1,2]; gam:=g[2,1]; del:=g[2,2];
    return [del*alp+gam*bet,-gam*alp,del*bet,-2*bet*alp,alp^2,-bet^2,2*del*gam,-gam^2,del^2];
end function;
```

(theoretically you need to coerce the result into the appropriate general linear group or matrix algebra for it to make sense), and secondly I wrote the function which computed the cohomology ($H^1$) of such a module using magma’s inbuilt cohomology calculator:

```magma
h1:=function(q)
    k:=GF(q);
    G:=SL(2,#k);
    M:=MatrixAlgebra(k,3);
    Y:=GModule(G,[M!end0(G.i):i in [1..#Generators(G)]]);
    D:=CohomologyModule(G,Y);
    return CohomologyGroup(D,1);
end function;
```

NOTE: `Generators(G)` returns a set, despite the fact that $G.i$ makes sense for $1 \leq i \leq \#\text{Generators}(G)$, so it could have been a sequence.

I could then write a loop which computed lots of $H^1$s and experimentally verified Lemma 2.48 of DDT:

```magma
for q:=3 to 100000000 by 2 do
    if IsPrimePower(q) then print q,h1(q); end if;
end for;
```

As expected, the output looks like

```
3  Full Vector space of degree 0 over GF(3)
5  Full Vector space of degree 1 over GF(5)
7  Full Vector space of degree 0 over GF(7)
9  Full Vector space of degree 0 over GF(3^2)
11 Full Vector space of degree 0 over GF(11)
```

(and then lots more zeros).