A case of Lloyd's theorem, and related computations.

Kevin Buzzard

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These are a summary of the arguments in one of the chapters of Lloyd Kilford's thesis. Note that Buzzard–Kilford generalises all this stuff far more. Last modified 19/08/2003.

1 Lloyd's theorem.

This is my notes on Lloyd's theorem that if $N = 4$ then the slopes of U_2 on modular forms of odd weight k and level 4 are $\{0, 2, 4, \ldots, k-1\}$, each with multiplicity one. Firstly note that for odd $k \geq 1$ the dimension of the space of Eisenstein series of weight k and level 4 is 2 for $k > 1$ and 1 for $k = 1$. The Eisenstein slopes are 0 and $k-1$. For $k \geq 3$ the non-ordinary Eisenstein series has no constant term so as an overconvergent form is cuspidal. Lloyd proves that the slopes of the cuspidal overconvergent forms are $\{2, 4, 6, \ldots\}$ each with multiplicity one, and this is equivalent to the classical statement above by Coleman's theorem. In the note below we actually prove that the slopes of the overconvergent modular forms are $\{0, 2, 4, 6, \ldots\}$ each with multiplicity one, this all boils down to the same thing.

Here is the proof. Firstly let's define $E = \sum_{a,b} q^{a^2+b^2}$ and $V = E(q^2)$ and $z = ((E/V)-1)/2$ (note: Lloyd uses $E/4$ not E, and he uses z for several things early on). Then

$$z = 2q - 8q^3 + 28q^5 - 80q^7 + 202q^9 - \ldots$$

is a function on $X_0(8)$. Let's also define j_4 by $j_4 = \left(\Delta(q)/\Delta(q^4)\right)^{1/3}$. Then one checks, for example, that $(j_4\Delta(q^4))^3$ as a weight 36 level 4 modular form is a cube of a weight 12 form (by examining power series coefficients), and hence j_4 is a modular function on $X_0(4)$. We have

$$j_4 = q^{-1} - 8 + 20q - 62q^3 + 216q^5 - 641q^7 + \ldots,$$

that

$$j_4 = (8z^2 - 8z + 2)/z,$$

and that

$$\frac{(j_4^2+256j_4+4096)^3}{j_4^4(j_4+16)} = j.$$

This last equation tells us that, for degree reasons, the map $j_4 : X_0(4) \to \mathbf{P}^1$ must be an isomorphism, and that j_4 has degree 2 as a function of z so $z : X_0(8) \to \mathbf{P}^1$ must also be an isomorphism.

The intrinsic "v" valuation on an elliptic curve with a canonical subgroup is the thing that should be less than $p/(p+1)$ for everything to work. Because E_4 is a lift of the 4th power of the Hasse invariant, and $j = (E_4)^3/\Delta$ we see that if E is an elliptic curve over $\mathbf{C}_2$ with a canonical subgroup, then $v(E) = v(j(E))/12$.

One checks easily that if $|j_4| > 2^{-4}$ then $|j| = |j_4|$. One also checks easily that if $|z| < 2$ then $|1/j| = 2|z|$. One deduces from all this that the region of $X_0(8)$ defined by $|z| \leq 1$ is in fact isomorphic to the region $|1/j| \leq 2$ in $X_0(1)$ and is hence the strict neighbourhood of the ordinary locus which goes by the name $X_0(1)_{\geq 2^{-1/12}}$. In particular this space will be stable under Hecke operators, and U will be compact.

It's easy to see that $U(z) = 0$ and hence $z = qF(q^2)$ for some $F \in \mathbf{Z}_2[[q]]$ and now it follows easily that $U(z^j) = 0$ if j is odd and $U(z^{2m}) = (qF^2(q))^m = U(z^2)^m$. A finite computation shows that $U(z^2) = 2z/(1+2z)^2$ (the left hand side is $U(E^2-2EV+V^2)/4E^2$ and the right hand side is $(EV-V^2)/E^2$ so it suffices to check that $U(E^2-2EV+V^2) = 4EV-4V^2$, which can be done by checking finitely many terms of the q-expansion). We deduce from this that if $U(z^j) = \sum_i a_{i,j}z^i$ then $F(X,Y) := \sum_{i,j} a_{i,j}X^iY^j = \sum_{m\geq 0}\left(2XY^2/(1+2X)^2\right)^m$, that is,

$$\begin{aligned} F(X,Y) &= \frac{1}{1-(2XY^2/(1+2X)^2)} \\ &= \frac{(1+2X)^2}{(1+2X)^2-2XY^2}. \end{aligned}$$

This rational function encodes all the arithmetic of the eigenvalues of U_2 in weight 0. Now V^kz^j, $j = 0,1,\ldots$ is a basis for the $2^{-1/12}$-overconvergent modular forms of weight-character (k,χ^k), with χ the non-trivial Dirichlet character of order 4, and $U(V^kz^j) = E^kU(z^k) = (E/V)^kV^kU(z^j) = (1+2z)^kV^kU(z^j)$ so the corresponding function F_k encoding the matrix of U in weight (k,χ^k) is $(1+2X)^kF(X,Y)$.

Note that $F(X,Y) \in \mathbf{Z}_2[[2X,Y^2]]$. This means two things: firstly, all odd columns are identically zero, and secondly the ith row is a multiple of 2^i. In particular if we throw away all the odd rows and columns, the resulting matrix has the ith row a multiple of 4^i. Now divide the ith row by 4^i and we get a formal infinite matrix which isn't compact any more (and perhaps not even continuous). Our goal is to show that the determinant of the top left hand n by n chunk of this matrix is a 2-adic unit, for all n (then it's just a matter of linear algebra). What is the function associated to the reduction mod 2 of this matrix? Well, we firstly replace X by $X/2$ to divide the $2i$th row by 4^i, giving us

$$\frac{(1+X)^{k+2}}{(1+X)^2 - XY^2},$$

we then replace Y^2 by Y (killing all the odd columns), giving

$$\frac{(1+X)^{k+2}}{(1+X)^2 - XY},$$

we then reduce mod 2 giving

$$\frac{(1+X)^{k+2}}{1+X^2+XY}$$

and we then have to throw away all the odd rows and we'll have the function representing the matrix we're interested in. On the function side of things, in characteristic 2, throwing away the odd rows amounts to sending a function G to $G - X(\partial G/\partial X)$ (giving a function of X^2 and Y) and then replacing X^2 by X. So let's do this: the derivative of $\frac{(1+X)^{k+2}}{1+X^2+XY}$ with respect to X is (and here is where we assume k is odd)

$$\frac{(1+X)^{k+1}(1+X^2+XY) + Y(1+X)^{k+2}}{1+X^4+X^2Y^2} \quad (1)$$

$$= \frac{(1+X)^{k+1}(1+X^2+Y)}{1+X^4+X^2Y^2} \quad (2)$$

and so the function we are interested in is

$$\frac{(1+X)^{(k+1)/2}(1+X+XY)}{1+X^2+XY^2}.$$

(Note that Kilford seems to mostly consider the case $k = 1$ even though he calls it F_k; also he subtracts 1 because he's not interested in the 0th row and column, as he deals only with cusp forms. Doing this recovers the function he has on p38 of his thesis). Our puzzle is now the following: for any $n \geq 1$ consider the above function as an element of $\mathbf{F}_2[X,Y]/(X^n, Y^n)$ and write it as $\sum_{i=0}^{n-1} f_i(X)Y^i$. Our goal is to prove that the $f_i(X)$ for $0 \leq i \leq n-1$ give an $\mathbf{F}_2$-basis for $\mathbf{F}_2[X]/(X^n)$. Kilford has a proof in his thesis but I will sketch another one, shown to me by Robin Chapman. Firstly observe that $(1+X)^{(k+1)/2}$ is a unit and so we can completely remove it from the expression for the function; this just multiplies all the f_i by the same unit so won't change whether they are linearly independent or not. Now the expansion for the function is $(1 + X + XY)/(1 + X^2 + XY^2)$, and for the same reason we can multiply the top by $(1 + X)$ and then divide top and bottom by $(1 + X)^2$ to deduce that the function we're interested in is

$$\left(1 + \frac{XY}{1+X}\right)(1 + T + T^2 + T^3 + \ldots)$$

with $T = \frac{XY^2}{1+X^2}$. Now it's easy to read off the f_i: we have $f_0 = 1$, $f_1 = X/(1+X)$, $f_2 = X/(1+X)^2$, $f_3 = X^2/(1+X)^3$, $f_4 = X^2/(1+X)^4$, $f_5 = X^3/(1+X)^5$, $f_6 = X^3/(1+X)^6$, $f_7 = X^4/(1+X)^7$ and so on. Now multiply by $(1+X)^n$ and show that the resulting vectors span $\mathbf{F}_2[X]/(X^n)$. I'll give the proof for $n = 8$: we have the eight vectors $(1+X)^7$, $X(1+X)^6$, $X(1+X)^5$, $X^2(1+X)^4$, $X^2(1+X)^3$, $X^3(1+X)^2$, $X^3(1+X)$ and X^4. The first vector takes care of the X^0 term, the second take care of X^7, the third takes care of X^1, the 4th takes care of X^6, the 5th takes care of X^2, the 6th of X^5, the 7th of X^3 and the 8th of X^4. It's clear that this works in general (well, maybe one should also check $n = 7$:-)). Done.

2 Other computations.

We have $z = ((E/V)-1)/2 = 2q - 8q^3 + 28q^5 - 80q^7 + 202q^9 - \ldots$ is a uniformiser on $X_0(8)$. Write $z = 2w$ with $w = ((E/V)-1)/4 = q - 4*q^3 + 14*q^5 - 40*q^7 + 101*q^9 - \ldots$. Set $i = 1/j$, set $j_2 = \Delta(q)/\Delta(q^2)$, set $i_2 = 1/j_2$. Then i_2 and j_2 are uniformisers on $X_0(2)$ because $j = 16777216j_2^{-2} + 196608j_2^{-1} + 768 + j_2$, that is,

$$j = \frac{(j_2 + 256)^3}{j_2^2}$$

and hence

$$i = \frac{i_2}{(256i_2 + 1)^3}.$$

Set $j_4 = (\Delta(q)/\Delta(q^4))^{1/3}$. Then j_4 is a uniformiser on $X_0(4)$; set $i_4 = 1/j_4$. Then

$$j_2 = \frac{j_4^2}{j_4 + 16}$$

and hence

$$i_2 = i_4 + 16i_4^2$$

Finally

$$j_4 = \frac{(8z^2 - 8z + 2)}{z}$$

and hence

$$i_4 = \frac{z}{(8z^2 - 8z + 2)}.$$

I was wondering what the relation between i_2 and w was. We see that

$$i_4 = \frac{w}{(4w-1)^2}$$

so

$$i_2 = \frac{w(4w+1)^2}{(4w-1)^4}$$