Automorphic forms over function fields.

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1 What is this document?

Ambrus Pal told me some facts about automorphic forms over function fields (and their relation to geometry of curves over finite fields). I need to write them down before I forget them! First though I need to write down the definitions of automorphic forms over function fields because I am a bit hazy on them. So here we go.

2 The definitions.

Note: there *will* be examples, but after the definitions it's natural to recall class field theory and to make several geometric and combinatorial remarks—then the examples can be explained in a much more conceptual manner—the theory guides the way so you're not just floundering around with matrices.

The definitions come from Borel-Jacquet in Corvallis. In the number field case we first make a definition of an automorphic form over a real reductive group, and then give an adelic definition and check that it basically specialises to the real reductive group definition in some sense. In the function field case we have to start with an adelic definition because there are no real places. So we let F now be a function field over a finite field k and write $\mathbf{A}$ for the adeles of F and so on. If G/F is connected reductive then a function on $G(\mathbf{A})$ is said to be *smooth* if it's locally constant. Let Z be the centre of G. We fix a (quasi-)character $\chi : Z(\mathbf{A})/Z(F) \to \mathbf{C}^\times$ and an open compact subgroup K of $G(\mathbf{A})$ (all places!). Define ${}^0V(\chi, K)$ to be the $\mathbf{C}$-valued functions $f : G(F)\backslash G(\mathbf{A})/K \to \mathbf{C}$ with $f(zx) = \chi(z)f(x)$, and such that f is *cuspidal*. The cuspidality condition (which I'm about to explain) is the reason for that little ^{0} by the V. There is no boundedness condition necessary in the function field case (in fact there are boundedness conditions mentioned in Godement-Jacquet (SLNM 260, p138) but they are actually implied by $f(zx) = \chi(z)f(x)$ and a compactness result below). The cuspidality condition is that for all $x \in G(\mathbf{A})$ the integral of $f(nx)dn$ vanishes, where the integral is taken over $N(F)\backslash N(\mathbf{A})$ (where N is the unipotent radical of any proper parabolic F-subgroup P of G). It suffices to check that the integral vanishes for P running through a set of reps of the conj classes of the proper maximal parabolic F-subgroups.

Note that these are *not* automorphic forms yet! There is still an "admissibility" condition to be imposed, which can perhaps either be thought of as some sort of finiteness condition or as some sort of condition of the form "I satisfy some differential equations". In fact in the version I worked out it seemed to say "Some sequence closely related to me satisfies a difference equation. Note also Proposition 4.5 of Borel-Jacquet, which says that in the presence of the other axioms for an automorphic form over a number field, $\mathfrak{z}$-finiteness (i.e. the condition about satisfying some differential equations, where here $\mathfrak{z}$ is the centre of the universal enveloping algebra) is equivalent to an admissibility condition at infinity.

Anyway, back to cusp forms. Harder proved that there was a compact subgroup C of $G(\mathbf{A})$ such that $f \in {}^0V(\chi, K)$ has support on a set of the form $Z(\mathbf{A})G(F).C$. In particular ${}^0V(\chi, K)$ is finite-dimensional. This is proved in Godement-Jacquet for GL_n (Lemma 10.9) and I know an explicit proof for GL_2 if $F = k(T)$, the field of fractions of a polynomial ring in one variable over a finite field (which I'll explain later on, when doing examples).

The admissibility condition that we need for a function to be an automorphic form is this: a function on $G(F)\backslash G(\mathbf{A})/K$ is *admissible* if for some (equivalently any! B-J 5.7) place v of F, the representation of $G(F_v)$ generated by f is admissible. Turns out that this implies $Z(\mathbf{A})$-finiteness (see the comment in B-J 5.8).

An automorphic form is an admissible map $G(F)\backslash G(\mathbf{A})/K \to \mathbf{C}$. A cusp form is a cuspidal automorphic form.

But perhaps surprisingly, all one needs to do to check that a cuspidal (that is, the integrals vanish) function on $G(F)\backslash G(\mathbf{A})/K$ is a cusp form, is to do is to check $Z(\mathbf{A})$-finiteness! This is B-J 5.9.

3 Adeles and geometry.

Ambrus explained these neat results to me, relating adeles to the geometry of the situation and to Weil groups. First let me mention Weil groups. The Weil group of F is a *subgroup* of the absolute Galois group of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ of F, rather more indicative of the non-arch local field case than the number field case. The algebraic closure of k gives a natural surjection $\mathrm{Gal}(F^{\mathrm{sep}}/F) \to \hat{\mathbf{Z}}$ and the Weil group is the pre-image of $\mathbf{Z}$ with the discrete topology in the usual way. Global class field theory in this setting tells us that the abelianisation of this Weil group is canonically isomorphic to the idele class group of F:

$$W_F^{\mathrm{ab}} = F^\times\backslash\mathbf{A}^\times.$$

Now let $\mathcal{C}$ denote the smooth projective geometrically connected curve over k corresponding to F and let $\hat{\mathcal{O}}$ denote the product of the integer rings at all the completions of F. If $\mathrm{Pic}(\mathcal{C})$ denotes the group isomorphism classes of invertible sheaves on $\mathcal{C}/k$ then we have a canonical isomorphism

$$\mathrm{Pic}(\mathcal{C}) = F^\times\backslash\mathbf{A}^\times/\hat{\mathcal{O}}^\times.$$

Furthermore there is a "degree" map $\deg : \mathbf{A}^\times \to \mathbf{Z}$ which has $F^\times$ and $\hat{\mathcal{O}}^\times$ in its kernel (the former by the product formula), and this map agrees with the usual degree map $\mathrm{Pic}(\mathcal{C}) \to \mathbf{Z}$, the kernel of which is of course $\mathrm{Pic}^0(\mathcal{C})$.

Ambrus explained to me a beautiful generalisation of the statement that $\mathrm{Pic}(\mathcal{C}) = F^\times\backslash\mathbf{A}^\times/\hat{\mathcal{O}}^\times$: if $n \geq 1$ then the isomorphism classes $\mathrm{Vec}_n(\mathcal{C})$ of the rank n vector bundles on $\mathcal{C}$ are in natural bijection with an adelic space:

$$\mathrm{Vec}_n(\mathcal{C}) = \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A})/ \mathrm{GL}_n(\hat{\mathcal{O}}).$$

Explicitly, the map from right to left is given by sending the element $g \in \mathrm{GL}_n(\mathbf{A})$ to the sheaf whose sections on the open $U \subseteq \mathcal{C}$ are $\{v \in F^n \mid vg \in \mathcal{O}_x^n\, \forall x \in U\}$.

4 Explicit double cosets in the case $\mathcal{C} = \mathbf{P}^1$.

If $F = k(T)$ and so $\mathcal{C}$ is just the projective line, then of course $\mathrm{Pic}(\mathcal{C}) = \mathbf{Z}$. Even better: Ambrus tells me that a vector bundle on $\mathbf{P}^1$ is isomorphic to a direct sum of line bundles! Hence any rank n vector bundle will be isomorphic to $\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_n)$. The a_i are integers and are uniquely determined up to order (one can see this by looking at dimensions of global sections of the vector bundle and all its twists by $\mathcal{O}(m)$ for m an integer). As a consequence we see that if we choose a degree 1 place $\infty \in \mathcal{C}$ and let ϖ be a uniformiser there, then we have a natural set of double coset representatives for $\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A})/ \mathrm{GL}_n(\hat{\mathcal{O}})$. For $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{Z}^n$ with $a_1 \leq a_2 \leq \cdots \leq a_n$ let $d_{\vec{a}} \in \mathrm{GL}_n(F_\infty)$ denote the diagonal matrix $\mathrm{diag}(\varpi^{a_1}, \varpi^{a_2}, \ldots, \varpi^{a_n})$. Let D be the set of all matrices $d_{\vec{a}}$ of this form, considered as a subset of both $\mathrm{GL}_n(F_\infty)$ and $\mathrm{GL}_n(\mathbf{A})$.

Lemma 1. *If $F = k(T)$ then $\mathrm{GL}_n(\mathbf{A}) = \coprod_{d \in D} \mathrm{GL}_n(F) d \,\mathrm{GL}_n(\hat{\mathcal{O}})$.*

Proof. Immediate from the above. □

In particular, because the coset generators can be taken to be supported at one place, we see that if $\mathbf{A}_f$ is the "finite adeles", that is, the restricted product $\prod_{v \neq \infty} F_v$, and $\hat{\mathcal{O}}_f$ is the corresponding product of the integer rings, then

Corollary 2. *If* $F = k(T)$ *then* $\mathrm{GL}_n(\mathbf{A}_f) = \mathrm{GL}_n(F)\,\mathrm{GL}_n(\hat{\mathcal{O}}_f)$*.* {2}

Proof. Take something on the left hand side, consider it in $\mathrm{GL}_n(\mathbf{A})$, write it as an element of $\mathrm{GL}_n(F)\gamma\,\mathrm{GL}_n(\hat{\mathcal{O}})$ with γ supported at infinity, and now restrict to the finite places. □

However I think we can do better! I *think* that weak approximation might say

Conjecture-Lemma 3. $\mathrm{SL}_n(F)$ *is dense in* $\mathrm{SL}_n(\mathbf{A}_f)$*.*

Proof. This might be standard? Platonov and Rapinchuk appear only to deal with the number field case though. □

Conjecture-Corollary 4. *If* $K_0 \subseteq \mathrm{SL}_n(\hat{\mathcal{O}}_f)$ *is a compact open subgroup then* $\mathrm{SL}_n(\mathbf{A}_f) = \mathrm{SL}_n(F)K$*.*

Proof. (assuming the conjecture-lemma) If $x \in \mathrm{SL}_n(\mathbf{A}_f)$ then xK is open so meets $\mathrm{SL}_n(F)$, and that's it. □

Conjecture-Corollary 5. *If* $F = k(T)$ *and if* $K \subseteq \mathrm{GL}_n(\hat{\mathcal{O}}_f)$ *is a compact open subgroup with* $\det(K) = \mathcal{O}_f^\times$ *then* $\mathrm{GL}_n(\mathbf{A}_f) = \mathrm{GL}_n(F).K$*.* {5}

Proof. (assuming the conjecture-lemma): By Corollary 2 (which is where we assume $F = k(T)$) we have $\mathbf{A}_f^\times = F^\times \mathcal{O}_f^\times$. So now given something in $\mathrm{GL}_n(\mathbf{A}_f)$ we can modify it on the left by an element of $\mathrm{GL}_n(F)$ and on the right by an element of K until it's in $\mathrm{SL}_n(\mathbf{A}_f)$, and the result follows from the previous conjecture-corollary. □

At the infinite place we deduce (do we need this?? Maybe for tree arguments) that if D denotes the set of $d_{(a_i)}$ as above, we have

Corollary 6. *If* $F = k(T)$ *then* $\mathrm{GL}_n(F_\infty) = \bigcup_{d \in D} \mathrm{GL}_n(F).d.\,\mathrm{GL}_n(\hat{\mathcal{O}}_\infty)$*.*

Proof. Same idea as in the proof of Corollary 2 but now just restrict to the infinite place. □

Corollary 7. $\mathrm{GL}_n(\mathbf{A}) = \mathrm{GL}_n(F)\,\mathrm{GL}_n(\hat{\mathcal{O}}_f)\,\mathrm{GL}_n(F_\infty)$*.*

Proof. Follows immediately from Corollary 2. □

Remark 8. This latter corollary just says that the Picard group becomes zero if you remove a point. Somehow in general probably the result is that vector bundles on an affine open in $\mathcal{C}$ are just some adelic quotient space where you remove the local components corresponding to the points you removed.

Conjecture-Corollary 9. *If* $F = k(T)$ *and* $K \subseteq \mathrm{GL}_n(\hat{\mathcal{O}}_f)$ *is compact and open with* $\det(K) = \hat{\mathcal{O}}_f$ *then* $\mathrm{GL}_n(\mathbf{A}) = \mathrm{GL}_n(F)K\,\mathrm{GL}_n(F_\infty)$*.* {9}

Proof. (assuming the conjecture-lemma) Follows immediately from conjectural corollary 5. □

5 The tree.

The tree $\mathcal{Z}$ itself (which I'm about to define) is a purely local object. It's an analogue of the upper half plane. If L is the field of fractions of any complete DVR then we can consider lattices in L^2 up to homothety and there you are—well, those are the vertices anyway. The edges are "adjacent" lattices. If the residue field is finite (which it always is, in our case) then the tree is locally finite in the sense that the vertices have finite degree. The group $\mathrm{GL}_2(L)$ acts naturally on L^2 and hence on the tree, and it's easy to check that the action is transitive.

Now set $L = F_\infty$, and let Z_∞ be the centre of $\mathrm{GL}_2(F_\infty)$. The tree $\mathcal{Z}$ associated to F_∞ has a canonical vertex $\mathcal{O}_\infty^2$, and the stabiliser of this vertex under the $\mathrm{GL}_2(F_\infty)$-action is just $Z_\infty.\mathrm{GL}_2(\mathcal{O}_\infty)$. Hence $\mathcal{Z}$ has vertices bijecting with $\mathrm{GL}_2(F_\infty)/Z_\infty.\mathrm{GL}_2(\mathcal{O}_\infty)$, and a similar calculation shows that it has edges bijecting with $\mathrm{GL}_2(F_\infty)/Z_\infty.\Gamma_0(\infty)$, where $\Gamma_0(\infty)$ is the obvious thing (upper triangular modulo the maximal ideal). There are "source" and "target" maps from the edges to the vertices; if $w = \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ with ϖ a uniformiser then the source map sends g to g and the target sends g to gw, which is well-defined because w normalises $Z_\infty.\Gamma_0(\infty)$. Note that the edges really are oriented here, and w induces a fixed-point-free bijection on the edges sending an edge to its "opposite", that is, the same edge but pointing the other way.

6 Quotients of the tree in the $\mathbf{P}^1$-case.

The point is that because $\mathrm{Pic}^0(\mathcal{C})$ is trivial in the $\mathbf{P}^1$ case, one can use those lemmas from an earlier section to relate quotients of the tree to adelic objects.

Here's how it works. Set $F = k(T)$ always in this section. Let $R = k[T]$ denote the functions on $\mathcal{C}\backslash\{\infty\}$, so $R = F \cap \hat{\mathcal{O}}_f$. The key point is

Lemma 10. *If* $F = k(T)$ *then* $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}) = \mathrm{GL}_2(R)\backslash \mathrm{GL}_2(\hat{\mathcal{O}}_f)\,\mathrm{GL}_2(F_\infty)$.

Proof. A corollary from an earlier section said

$$\mathrm{GL}_2(\mathbf{A}) = \mathrm{GL}_2(F)\,\mathrm{GL}_2(\hat{\mathcal{O}}_f)\,\mathrm{GL}_2(F_\infty)$$

and the result follows from this and the second isomorphism theorem or whatever it's called: $A\backslash AB = (A \cap B)\backslash B$. □

{11}

Conjecture-Lemma 11. *If* $F = k(T)$ *and* $K_f \subseteq \mathrm{GL}_2(\hat{\mathcal{O}}_f)$ *is compact and open with* $\det(K_f) = \hat{\mathcal{O}}_f^\times$ *then* $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(A) = \Gamma_{K_f}\backslash K_f\,\mathrm{GL}_2(F_\infty)$ *where* $\Gamma_{K_f} = K_f \cap \mathrm{GL}_2(F)$ *(where the intersection is in* $\mathrm{GL}_2(\mathbf{A}_f)$ *but* Γ_{K_f} *is thought of as a subset of* $\mathrm{GL}_2(F)$ *embedded diagonally).*

Proof. (Assuming the conjecture-lemma) Again it's just the second isomorphism theorem, plus conjecture-corollary 9. □

Corollary 12. *If* $\mathcal{Z}$ *is the tree then* $\mathrm{GL}_2(R)\backslash\mathcal{Z} = \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A})/\mathrm{GL}_2(\hat{\mathcal{O}}).Z_\infty$.

Proof. The lemma shows that the right hand side is $\mathrm{GL}_2(R)\backslash \mathrm{GL}_2(F_\infty)/GL_2(\mathcal{O}_\infty).Z_\infty$ so we're done by the description of $\mathcal{Z}$ given in the previous section. □

Conjecture-Corollary 13. *If* $K = K_f.K_\infty$ *is compact open with* $\det(K) = \hat{\mathcal{O}}_f^\times$ *then*

$$\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A})/K.Z_\infty = \Gamma_{K_f}\backslash \mathrm{GL}_2(F_\infty)/K_\infty.Z_\infty.$$

Proof. (Assuming the Conjecture-lemma) Immediate by Conjecture-Lemma 11. □

Conjecture-Corollary 14. *If furthermore* K_∞ *is* $\mathrm{GL}_2(\mathcal{O}_\infty)$ *or* $\Gamma_0(\infty)$ *then in the setting above, the double quotient space is the quotient by* Γ_{K_f} *of either the vertices or the edges of the tree.*

Proof. Immediate. □

7 Examples

Finally.

7.1 GL_1.

Let me do the GL_1 case first. If $G = \mathrm{GL}_1$ and K is $\hat{\mathcal{O}}^\times$ then $G(F)\backslash G(\mathbf{A})/K$ is $\mathrm{Pic}(\mathcal{C})$ which has a finite subgroup $\mathrm{Pic}^0(\mathcal{C})$ and quotient isomorphic to $\mathbf{Z}$ via the degree map. I guess that more generally if K is any compact open then there will still be a degree map and the kernel will be some finite ray-class-group thing. That space ${}^0V(\chi, K)$ is going to have dimension either 0 or 1 because cuspidality is no condition at all (no proper parabolics) and $Z = G$ so saying what happens on Z is saying everything; the space will be 1-dimensional, and contain χ, iff $\chi(K) = \{1\}$.

An arbitrary function on $G(F)\backslash G(\mathbf{A})/K$ generates a smooth representation of $G(F_v)$ and this representation is fixed by K_v and hence admissible iff it's finite-dimensional. So admissibility is visibly equivalent to Z-finiteness in this case. In the case $\mathcal{C} = ¶^1$ and $K = \hat{\mathcal{O}}^\times$, the double coset space is isomorphic to $\mathbf{Z}$ and admissibility is equivalent to "the induced function on $\mathbf{Z}$ is of the form $n \mapsto a_n$ with the a_n satisfying a linear recurrence relation", that is, a difference equation, which is some kind of discrete version of a differential equation. But in the general case (F and K arbitrary) a similar thing is going on: if you choose a place ∞

Back to the general case (F and K arbitrary). A representation of $G(\mathbf{A})$ shows up as a subset of the space of cuspidal automorphic forms iff

7.2 GL_2 when $F = k(T)$.

Well, let's do PGL_2 because it has no centre. Let K_0 be $\mathrm{PGL}_2(\hat{\mathcal{O}})$, let K be a finite index open subgroup of K_0, and let's consider maps $G(F)\backslash G(\mathbf{A})/K \to \mathbf{C}$. First let's try and understand the double coset space. Let's let $K = K_0$ for a minute. Then if $G = \mathrm{GL}_2$ then the double coset space is just rank 2 vector bundles up to isomorphism, so (because we're doing the projective line) it's pairs $a \leq b$ of integers. Now quotient out by the centre and we see that if $G = \mathrm{PGL}_2$ then we're left with $b - a$ so it's $\mathbf{Z}_{\geq 0}$, with coset representatives given by $\gamma_n := \left(\begin{smallmatrix} \varpi^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for ϖ any uniformiser concentrated at any random degree 1 place, for example. One can alternatively think of the double coset space as the quotient of the tree T by $\mathrm{GL}_2(R)$, where $R = k[T]$ is functions on the affine line, and the "distance from the fixed point" map gives the natural isomorphism between this quotient space and $\mathbf{Z}_{\geq 0}$.

If K is some open subgroup of K_0 and $G = \mathrm{PGL}_2$ and we write $K_0 = \coprod k_i K$ then the $\gamma_n k_i$ will certainly contain a set of representatives for the double coset space $G(F)\backslash G(\mathbf{A})/K$.

Harder's theorem says that anything cuspidal has compact support, and hence in this case finite support. Let's see a proof of this in this case. Firstly, what do these integrals mean? I think that $\mathbf{A} = F + \hat{\mathcal{O}}$; I can prove it by induction on number of poles, as it were, it's easy because $F = k(T)$ (it wouldn't surprise me if it were true in general though!): all I have to do is to prove that if v is a place, if $n > 0$ and if $u_v \in F_v$ is a local unit and $\varpi_v \in F_v$ is a uniformiser, then there's $f \in F$ such that $f = u_v \varpi_v^{-n} + \cdots$ in F_v and furthermore such that f is integral everywhere else. But one just writes down the polynomial, I guess, which makes it work. OK so in that case we have $\mathbf{A} = F + \hat{\mathcal{O}}$ and $F \cap \hat{\mathcal{O}}$ is k so we can think of the integral defining cuspidality as an integral over $\hat{\mathcal{O}}/k$ and because k is finite we may as well just integrate over $\hat{\mathcal{O}}$ and demand that this integral vanish.

Now the key observation is that $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi^a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi^a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \varpi^a t \\ 0 & 1 \end{smallmatrix}\right)$ and for $f : G(F)\backslash G(A)/K \to \mathbf{C}$ to be cuspidal (general K in K_0 now) we need that the integral of $f(nx)dn$ vanishes, and hence that the integral $f(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)x)dt$ vanishes as t runs through $\hat{\mathcal{O}}$. Now fix K and coset reps k_i, and then choose a so large that $\left(\begin{smallmatrix} 1 & \varpi^a t \\ 0 & 1 \end{smallmatrix}\right)k_i$ is a subset of K; you can't do this. Doh. You can if $K = K_0$ though and this would have proved that the integral vanishes for a sufficiently large.

Vague aim: to see that the Steinberg is all to do with functions on the edges of the graph. **Todo possibly: write down explicit proof of Harder's theorem in this case.** KMB, originally written 12/9/07, last edited 19/11/07.