Automorphic forms for GL_1.

Kevin Buzzard

February 7, 2012

Last modified 13/04/2007.

1 Background.

If G is a connected reductive algebraic group over $\mathbf{Q}$ and K_∞ is a maximal compact subgroup of $G(\mathbf{R})$ then there is a notion of an automorphic form for the pair (G, K_∞): it's a certain type of smooth function on $G(\mathbf{A})$. I once worked out exactly what it said for $G = \mathrm{GL}_1/\mathbf{Q}$ and $K_\infty = \{\pm 1\}$, the unique maximal compact, and to my surprise there were examples which involved logs: indeed it was *not* the case that every automorphic form was a linear combination of Grössencharacters, which surprised me a little. On the other hand there was a rather brutal way to fix this: one could demand that every automorphic form had a central character. This same fix would work for automorphic forms on GL_1 over a number field too. However there is a slightly more subtle fix that people sometimes choose and the purpose of this note is to work out what happens if one uses this slightly more subtle fix. The more subtle fix is to let Z denote the maximal $\mathbf{Q}$-split torus in the centre of G and to only demand that an automorphic form has a character for $Z(\mathbf{A})$, something which is typically smaller than the adelic points of the centre of G. Indeed, if G is GL_1 over a number field then $Z(\mathbf{A})$ is a lot smaller than $G(\mathbf{A})$ in general. So let F be a number field and let's work out what automorphic forms for $G := \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_1)$ with "a character for Z" are.

2 Definitions.

Let F be a number field (we avoid K because K will be a compact subgroup). Rather than work with an arbitrary abelian G, let us set $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_1)$. Let $K_\infty \subset G(\mathbf{R}) = \prod_{v|\infty} F_v^\times$ be the unique maximal compact, namely the product of ± 1s at the real places and S^1s at the complex places, and now let's recall the definition of an automorphic form in this situation (G assumed abelian).

An automorphic form for (G, K_∞) is a smooth function $f : \mathbf{A}_F^\times \to \mathbf{C}$ (that is, a continuous function on $\mathbf{A}_F^\times = (\mathbf{A}_F^f)^\times \times F_\infty^\times$ which, if viewed as a function of two variables $x \in (\mathbf{A}_F^f)^\times$ and $y \in F_\infty^\times$, is C^∞ in y for fixed x and is locally constant in x for fixed y), such that

(a) $f(\gamma g) = f(g)$ for all $\gamma \in F^\times$,

(b) f is constant on cosets of K_f for some open compact subgroup of the finite adeles, and is K_∞-finite,

(c) There is an ideal of finite codimension of the universal enveloping algebra of $F_\infty^\times$ which annihilates f (this is an "admissibility" condition at infinity: see Proposition 4.5 of Borel-Jacquet), and

(d) a growth condition: for each fixed $x \in (\mathbf{A}_F^f)^\times$ the induced function $y \mapsto f(xy)$ on $F_\infty^\times$ is slowly increasing, that is, is a function on a product of some $\mathbf{R}^\times$s and $\mathbf{C}^\times$s which is, vaguely, bounded above by polynomials in the "norm" function $\max_{v|\infty}\left\{\max\{|y_v|, |y_v|^{-1}\}\right\}$.

We will furthermore impose

(e) There is a (quasi-)character χ of $\mathbf{A}_{\mathbf{Q}}^{\times}$, regarded as a subgroup of $\mathbf{A}_F^{\times}$, such that $f(zg) = f(g)$ for all $z \in \mathbf{A}_{\mathbf{Q}}^{\times}$.

3 Reduction of the calculation to places at infinity.

By K_∞-finiteness, we can decompose such a function into a sum of functions which are actually K_∞-eigenvectors. So let us now fix a quasicharacter χ of $\mathbf{A}_{\mathbf{Q}}^{\times}$ (trivial on $\mathbf{Q}^{\times}$) and a character ρ of K_∞, and let us consider the space of functions satisfying (a)–(e) above and furthermore that $f(gk_\infty) = \rho(k_\infty)f(g)$ for all $k_\infty \in K_\infty$. What can we say about such functions? Well, they are continuous functions on $\mathbf{A}_F^{\times}$ and we understand how they behave on cosets of $F^{\times}$, K_∞, $\mathbf{A}_{\mathbf{Q}}^{\times}$ and K_F. What is left?? Surely not much! For simplicity, let's replace $\mathbf{A}_{\mathbf{Q}}^{\times}$ by the smaller subgroup $\mathbf{R}_{>0}$ (embedded diagonally at the infinite places).

Lemma 3.1. *The norm map $\mathbf{A}_F^{\times} \to \mathbf{R}_{>0}$ induces a continuous map $F^{\times}\backslash\mathbf{A}_F^{\times} \to \mathbf{R}_{>0}$ with compact kernel.*

Remark 3.1.1. This norm map $F^{\times}\backslash\mathbf{A}_F^{\times} \to \mathbf{R}_{>0}$ is a global analogue of the norm map $L^{\times} \to \mathbf{R}_{>0}$ where L is a local field, and the kernel is a global analogue of the units of L.

Proof. This is, for example, Theorem 2.8 of Neukirch's book on class field theory. The statement incorporates finiteness of class group and the units theorem. □

No doubt one deduces from this that the growth condition (d) follows from (e). On the other hand this does seem to indicate that there is something left, and we'll have to hope that the admissibility condition (c) brings the space down to a finite-dimensional one.

So let's get our hands dirty and do the calculation. More specifically, let's now fix χ and ρ, and a finite index ideal I of the universal enveloping algebra at infinity, and let's consider the space of all automorphic forms transforming via χ and ρ and annihiliated by this ideal. Is this space finite-dimensional? By finiteness of the class group, we are reduced to trying to understand the space of functions $f : (F_\infty^{\times})^\circ \to \mathbf{C}$ which satisfy

(a) $f(ug) = f(g)$ for all u in some finite index subgroup of the group of totally positive global units,

(b) $f(gk_\infty) = \rho(k_\infty)f(g)$ for all $k_\infty \in (S^1)^{r_2}$ (embedded via the complex places),

(c) $if = 0$ for all $i \in I$,

(d) growth condition,

(e) $f(rg) = \chi(r)f(g)$ for all $r \in \mathbf{R}_{>0}$ (embedded diagonally in $(F_\infty^{\times})^\circ$)).

I guess that general theorems tell me that this space is finite-dimensional (even without condition (e)) but on the other hand I am wondering whether (a)–(e) actually tell me that f is a finite sum of characters.

4 Final analysis.

(Wasn't that a film?). Anyway, the first thing to do is to note that $\mathbf{C}^{\times} = S^1 \times \mathbf{R}_{>0}$ so we can remove condition (b) by taking absolute values, and the second thing to note is that condition (e) just involves removing another degree of freedom. Now if $n = r_1 + r_2 - 1$ then by a change of variables we see that we have to understand the C^∞ functions $F : \mathbf{R}^n \to \mathbf{C}$ such that

(a) $F(v + z) = F(v)$ for all $z \in \mathbf{Z}^n$

(c) $iF = 0$ for all $i \in I \subseteq \mathbf{C}[\partial/\partial x_j, 1 \leq j \leq n]$

(d) growth condition (F grows at most exponentially)

Now of course we see that (a) implies (d), so we can drop it, and we need to study periodic solutions to differential equations and see if such things are finite-dimensional. In fact (a) says that we may as well think of F as a function on $\mathbf{T}^n$, where $\mathbf{T}$ is the circle group, and by the theory of the Fourier transform we see that F is determined by its Fourier coefficients, which will be in $\ell^2(\mathbf{Z}^n)$. Now if $(c_t)_{t\in\mathbf{Z}}$ are the Fourier coefficients of a C^∞ function in one variable then the Fourier coefficients of its derivative are just (itc_t), so what we deduce from (c) is the existence of a finite index ideal $I = (f_1, \ldots, f_r) \subseteq \mathbf{C}[X_1, \ldots, X_n]$ with the property that $f_j(it_1, \ldots, it_n)c_{t_1,\ldots,t_n} = 0$ for all j and for all vectors $\vec{t} \in \mathbf{Z}^n$. Because the ideal is finite index, the subset of $\mathbf{C}^n$ where all the f_j vanish is finite, and hence all but finitely many of the Fourier coefficients must vanish! Hence F is a finite linear combination of characters which is what we were after, I think.