Automorphic forms for compact groups.

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February 7, 2012

Last modified 16/04/2007.

1 The key players.

By "compact" I of course mean G connected reductive over $\mathbf{Q}$ such that $G(\mathbf{R})$ is compact. Examples of such G include things like norm 1 elements of arbitrary definite quaternion algebras over $\mathbf{Q}$, or unitary groups. We could work more generally with connected reductive $G/\mathbf{Q}$ with the property that the maximal split (over $\mathbf{Q}$) torus in the centre of G has the property that its base extension to $\mathbf{R}$ is a maximal split torus in $G/\mathbf{R}$ (and not just in the centre of $G/\mathbf{R}$). That mouthful is equivalent to $\{1\}$ being an arithmetic subgroup of G, and all compact groups satisfy it. In particular note that $G(\mathbf{R})$ being compact implies that all arithmetic subgroups of $G(\mathbf{Q})$ are finite. It does not however imply that $G(\mathbf{Q})$ itself is finite—take for example the norm 1 elements in the hamilton quaternions over $\mathbf{Q}$; there are infinitely many solutions in $\mathbf{Q}$ to $a^2+b^2+c^2+d^2=1$ (but only finitely many in $\mathbf{Z}$).

Usually in the definition of an automorphic form for G we have to choose a maximal compact subgroup K_∞ of $G(\mathbf{R})$ but here we are forced to choose $K_\infty := G(\mathbf{R})$. In the "classical" definition we also have to choose an arithmetic subgroup Γ of $G(\mathbf{Q})$ and we recall that Γ will always be finite.

2 Classical definitions.

A smooth function $f : G(\mathbf{R}) \to \mathbf{C}$ is an automorphic form if

(a) $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$

(b) f is right K_∞-finite (that is, $G(\mathbf{R})$-finite!)

(c) Some condition involving the universal enveloping algebra

(d) some boundedness condition.

Let me now show that (c) and (d) are in fact automatically implied by (a) and (b) in the case $G(\mathbf{R})$ compact.

By K_∞-finiteness, we can decompose a function satisfying (a) and (b) above into a sum of functions f_ρ such that the K_∞ span of f_ρ is a direct sum of finitely many copies of ρ, a fixed irreducible representation of K_∞. So let us fix an n-dimensional irreducible complex (unitary Hilbert space) representation $\rho : K_\infty \to \mathrm{GL}(V)$ and let us consider the following two spaces:

(i) $A(\rho, \Gamma)$, the functions satisfying (a), (b) above, and

(ii) $A(V, \Gamma)$, the smooth functions $f : G(\mathbf{R}) \to V$ such that

(a') $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$

(b') $f(gk) = \rho(k^{-1})f(g)$ for all $k \in K_\infty$.

Note that for these to really be automorphic forms I need to add conditions (c') and (d') above; but let me not do this for a minute.

Question: how are (i) and (ii) related??

I can answer this now. Shrinking Γ only makes the spaces bigger, so WLOG $\Gamma = 1$. In this case the Peter-Weyl theorem tells us that if V^* is the dual of V then $A(\rho, \{1\})$ is abstractly isomorphic to $V^* \otimes V$ (as a representation of $G(\mathbf{R}) \times G(\mathbf{R})$!) and hence has dimension n^2. The subspace

$A(\rho,\Gamma)$ is simply $V^*\otimes V^\Gamma$. On the other hand an element of $A(V,\Gamma)$ is clearly determined by its value at 1, which is in V^Γ, so we deduce that the natural map $\overline{V}\otimes A(V,\Gamma)\to A(\rho,\Gamma)$ sending $v\otimes F$ to the function $g\mapsto(v,F(g))$ is an isomorphism and that $A(V,\Gamma)$ and $A(\rho,\Gamma)$ are all finite-dimensional. I've finally done that calculation!

Now let's get back to the missing conditions (c) and (d). The reason we don't need (d) is that the norm on $G(\mathbf{R})$ is bounded, but any continuous function on a compact group is also bounded, so (d) is automatic. As for (c), note that the universal enveloping algebra actually *acts* on $A(\rho,\{1\})$! For if $f\in A(\rho,\{1\})$ then (again by Peter-Weyl) we can write f as a finite linear combinations of functions $\phi_{v,w}$ for $v,w\in V$, where $\phi_{v,w}(g)=(v,gw)$. Now if and X is in the Lie algebra of $G(\mathbf{R})$ then $(Xf)(g)=(d/dt)f(ge^{Xt})|_{t=0}$ by definition, so $(X\phi_{v,w})(g)=(d/dt)(v,ge^{Xt}w)|_{t=0}=(v,gXw)$ and hence $X\phi_{v,w}=\phi_{v,Xw}$ (note that the Lie algebra of G also acts on V). So everything which is K_∞-finite is automatically $Z(\mathfrak{g})$-finite too.

3 Adelic definitions.

A (scalar-valued) automorphic form for (G,K_∞) is a smooth function $f:G(\mathbf{A})\to\mathbf{C}$ (that is, a continuous function on $G(\mathbf{A})=G(\mathbf{A}^f)\times G(\mathbf{R})$ which, if viewed as a function of two variables $x\in G(\mathbf{A}^f)$ and $y\in G(\mathbf{R})$, is C^∞ in y for fixed x and is locally constant in x for fixed y), such that

(a) $f(\gamma g)=f(g)$ for all $\gamma\in G(\mathbf{Q})$,

(b) f is constant on cosets of K_f for some open compact subgroup of $G(\mathbf{A}^f)$, and is K_∞-finite.

The other conditions (finite under centre of universal enveloping algebra and boundedness) are implied, as above. We can also fix a type, that is, a finite-dimensional irreducible complex (unitary Hilbert space) representation $\rho:K_\infty\to\mathrm{GL}(V)$, and consider

(i) $A(\rho,K_f)$, the functions satisfying (a) above, and whose K_∞-span is isomorphic to a finite direct sum of copies of ρ, or

(ii) $A(V,K_f)$, the smooth functions $f:G(\mathbf{A})\to V$ such that

(a') $f(\gamma g)=f(g)$ for all $\gamma\in G(\mathbf{Q})$

(b') $f(gk)=f(g)$ for all $k\in K_f$ and $f(gk)=\rho(k^{-1})f(g)$ for all $k\in K_\infty$.

Again these are spaces of automorphic forms, and $A(V,K_f)\otimes V=A(\rho,K_f)$. I kind of prefer $A(V,K_f)$ for this reason.

4 Borel's finiteness theorem.

Note that axioms (a') and (b') imply that $f\in A(V,K_f)$ is determined by its values on a set of representatives for $G(\mathbf{Q})\backslash G(\mathbf{A})/K_fG(\mathbf{R})$. A theorem of Borel (valid I think for arbitrary linear algebraic $G/\mathbf{Q}$) says that this double coset space is finite; let $\{g_\alpha\}_{\alpha\in C}$ be a set of representatives, and for $\alpha\in C$ set $\Gamma_\alpha=G(\mathbf{Q})\cap g_\alpha(G(\mathbf{R})\times K_f)g_\alpha^{-1}$. Then Γ_α is known to be a finite group (as it's an arithmetic subgroup of $G(\mathbf{Q})$), and if $\Delta_\alpha=g_\alpha^{-1}\Gamma_\alpha g_\alpha$ is regarded as a subgroup of $G(\mathbf{R})$ via the projection then we see that evaluating $f\in A(V,K_f)$ at $(g_\alpha)_{\alpha\in C}$ gives us an isomorphism $A(V,K_f)\to\oplus_{\alpha\in C}V^{\Delta_\alpha}$. In particular we see again that $A(V,K_f)$ is finite-dimensional.

5 Switching the action to ℓ.

Let's write U for K_f.

Say $A(V,U)$ is "the classical model" for the V-valued automorphic forms.

The "algebraic model" for $A(V,U)$ is given by restricting $f\in A(V,U)$ to $f_0:G(\mathbf{A}_f)\to V$; then f_0 satisfies $f_0(\gamma gu)=\gamma.f(g)$ for $\gamma\in G(\mathbf{Q})\subseteq G(\mathbf{R})$ and $u\in U$ (we're thinking here of $G(\mathbf{Q})$ as living in $G(\mathbf{R})$ and hence acting on V). Note now that V only has to be a representation of $G(\mathbf{Q})$ and in particular we can think of it as being defined over a number field. Given f_0 one can recover f (if V really is a rep of all of $G(\mathbf{R})$ via the rule $f(g_fg_\infty)=g_\infty^{-1}f_0(g_f)$.

The "ℓ-adic model" of $A(V,U)$ relies on the fact that the representation of $G(\mathbf{Q})$ can be naturally extended to one of $G(\mathbf{Q}_\ell)$, or perhaps just to one of U_ℓ, the compact open of U "at ℓ". Given $f_0 : G(\mathbf{A}_f) \to V$ define $f_\ell : G(\mathbf{A}_f) \to V$ by $f_\ell(g) = g_\ell^{-1} f_0(g)$. Then $f_\ell(\gamma g u) = u_l^{-1} f_\ell(g)$ and we go backwards (when V is a representation of all of $G(\mathbf{Q}_\ell)$) by $f_0(g) = g_\ell . f_\ell(g)$.

Hecke operators: say $U\eta U = \coprod \eta_i U$. Write $T = [U\eta U]$.

In the classical model we have $(Tf)(g) = \sum f(g\eta_i)$. This is independent of the choice of η_i and one checks Tf satisfies the same equation as f.

In the algebraic model we have $(Tf_0)(g) = \sum f_0(g\eta_i)$.

In the ℓ-adic model we have $(Tf_\lambda)(g) = \sum (\eta_i)_\ell . f_\lambda(g\eta_i)$

so for $\eta_\ell = 1$ there is no trouble, but for $\eta_\ell \neq 1$ one has to ensure that the representation of V naturally extends to one where η_ℓ acts too.

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