Some notes on principal series representations.

Kevin Buzzard

September 7, 2016

1 Definitions.

(Written 3/6/04; typo fixed 4/4/6)

Let K be a finite extension of $\mathbf{Q}_p$, let q denote the cardinality of the residue field, let $|.| : K^\times \to \mathbf{C}^\times$ be the map sending a uniformiser to $1/q$. Fix an integer $n \geq 1$. These are some notes about the principal series representations of $G := \mathrm{GL}_n(K)$.

If $\chi_1, \chi_2, \ldots, \chi_n$ are continuous group homomorphisms $K^\times \to \mathbf{C}^\times$ then there is an obvious representation of the Borel B of upper triangular matrices in G associated to this collection: namely the representation $\chi : B \to \mathbf{C}^\times$ defined by $\chi(b) = \prod_{i=1}^n \chi_i(b_{ii})$. Define $I(\chi)$ to be the normalised induction of χ from B to G. We recall explicitly what this is in this case: define a function $\delta : B \to \mathbf{C}^\times$ by

$$\delta(b) = \prod_{1 \leq i < j \leq n} |b_{ii}/b_{jj}| = \prod_{i=1}^n |b_{ii}|^{n+1-2i}.$$

Then

$$I(\chi) = \{f : G \to \mathbf{C} | f(bg) = \chi(b)\delta(b)^{1/2} f(g)\}$$

where the functions are locally constant, $b \in B$ and $g \in G$. Define a G-action on this space by $gf(h) = f(hg)$. The normalisation constant I think screws up Frobenius reciprocity a bit but it makes some other things work much better.

It turns out that $I(\chi)$ is irreducible if $\chi_i \neq \chi_j|.|$ for all i, j (and I think this might be iff). If $I(\chi)$ and $I(\chi')$ are irreducible then they are isomorphic iff the characters χ_i and χ'_i are the same after possible reordering. An example of a reducible case is when $n > 1$ and $\chi = \delta^{-1/2}$; for example if $n = 2$ this corresponds to $\chi_1 = |.|^{-1/2}$ and $\chi_2 = |.|^{1/2}$. In this case $I(\chi)$ is just the functions $B\backslash G \to \mathbf{C}$ and the constant functions are a G-invariant one-dimensional subspace. It turns out that if $n = 2$ then the quotient is irreducible (the Steinberg representation) but if $n > 2$ then one also has to worry about functions which are invariant under other parabolics containing B and the situation is more complicated (but still well-understood). I think that there is a unique generic irreducible subquotient of $I(\chi)$ but I forget why I think this, and I think it's the Steinberg representation.

2 Duals.

The dual $\widetilde{I(\chi)}$ or $I(\chi)^\vee$ of $I(\chi)$ is just the smooth vectors in the algebraic dual space to $I(\chi)$. Because we are using normalised induction, the dual of $I(\chi)$ is just $I(\tilde{\chi})$, where $\tilde{\chi}$ denotes the dual of χ. Becuase χ is 1-dimensional, its dual is just its inverse, so $\widetilde{I(\chi)} = I(\chi^{-1})$. The reason for all this is that the duality is explicitly given by an integral; given elements of $I(\chi)$ and $I(\chi^{-1})$ the product is a function $G \to \mathbf{C}$ that transforms in a certain way related to Haar measure on the Borel; but there is some kind of twisted integral ("Haar measure on $B\backslash G$"; Theorem 1.21 of Bernshtein-Zelevinskii Russian Math Surveys) that takes as input exactly such a function and gives out a number; this gives the duality.

3 Twisting.

Let ψ be a continuous group homomorphism $K^\times \to \mathbf{C}^\times$. Then we can twist the representation $I(\chi)$ by the 1-dimensional representation $\psi \circ \det$ of G. If we think of $I(\chi) \otimes (\psi \circ \det)$ as being the set $I(\chi)$ but with a twisted action defined by $gf(h) = \psi(\det(g))f(hg)$, and, for $f \in I(\chi) \otimes (\psi \circ \det)$, we define $\tilde{f} : G \to \mathbf{C}$ by $\tilde{f}(g) = f(g)\psi(\det(g))$, then we check easily that $\tilde{f}(bg) = f(bg)\psi(\det(bg)) = \chi(b)\delta(b)^{1/2}\psi(\det(b))\tilde{f}(g)$ and hence $\tilde{f} \in I(\chi.(\psi \circ \det))$. Moreover, recalling that $f \in I(\chi) \otimes (\psi \circ \det)$, so the action of G on f is defined by $(gf)(h) = f(hg)\psi(\det(h))$, we see that $\widetilde{gf} = g\tilde{f}$, so we have proved that

$$I(\chi) \otimes (\psi \circ \det) = I(\chi.(\psi \circ \det)).$$

4 Hecke actions.

Let ϖ be a uniformiser of K. Say $\chi = (\chi_i)$ and all the χ_i are unramified, and $\chi_i(\varpi) = \alpha_i$. Let us also assume that $I(\chi)$ is irreducible, although I don't know whether that's necessary (probably not). If $0 \leq m \leq n$ then define $\gamma_m \in G$ to be the diagonal matrix $\mathrm{diag}(\varpi, \varpi, \ldots, \varpi, 1, 1, \ldots, 1)$ where there are m ϖs. Define $G_0 := \mathrm{GL}_n(\mathcal{O})$ where $\mathcal{O}$ is the integers in K. Recall that the Hecke algebra associated to $G_0\backslash G/G_0$ is commutative and if T_m is the Hecke operator $[G_0\gamma_m G_0]$ then this Hecke algebra can be formed by taking the polynomial ring in the T_m, $1 \leq m \leq n$, and then inverting T_n. This Hecke algebra acts on the 1-dimensional space of K-invariants in $I(\chi)$ (recall I am assuming the χ_i are unramified) and if σ_m is the mth symmetric polynomial in the α_i then the eigenvalue of T_m on this 1-dimensional space is $q^{\frac{m(n-m)}{2}}\sigma_m$. For example T_1 acts as $q^{(n-1)/2}(\alpha_1 + \alpha_2 + \ldots + \alpha_n)$ and T_n acts as $\alpha_1\alpha_2\ldots\alpha_n$. In his thesis, Russ Mann works out how these Hecke operators act on $I(\chi)^M$ for certain other compact open subgroups M (M for "mirahoric", I think).

5 Outer automorphisms.

The group G has an outer automorphism ι, sending g to g^{-t}, the inverse of the transpose of g. If π is a representation of G then so is $\pi \circ \iota$, and let's call this representation $\iota(\pi)$. If $I(\chi)$ is irreducible and unramified (with $\chi_i(\varpi) = \alpha_i$ as usual), here's a way of working out $\iota(I(\chi))$: an unramified representation is determined by the associated representation of the unramified Hecke algebra. Because G_0 is preserved by inverse and transpose, we see that if $G_0 g G_0 = \coprod G_0 g_i$ then $G_0 g^{-t} G_0 = \coprod G_0 g_i^{-t}$ and hence if $[G_0 g G_0]$ acts on $v \in I(\chi)^{G_0}$ as multiplication by λ then $[G_0 g^{-t} G_0]$ acts on $v \in \iota(I(\chi))^{G_0}$ as multiplcation by λ as well. Because $[G_0 \gamma_m^{-t} G_0] = [G_0 \gamma_{n-m}/\varpi G_0] = [G_0 \gamma_{n-m} G_0][G_0 \varpi^{-1} G_0]$ and T_n has eigenvalue $\prod_i \alpha_i$, we see that the eigenvalue for T_m on $\iota(I(\chi))$ is $\prod_i \alpha_i^{-1}$ times the eigenvalue of T_{n-m} on $I(\chi)$, which is the smae as the eigenvalue of T_m on $I(\chi^{-1})$. I guess this is enough to conclude that $\iota(I(\chi)) = I(\chi^{-1})$ for unramified irreducible principal series, although it may be easy to work out what's going on in the general irreducible case as well.

6 Normalisations for Local Langlands.

Say $\pi := I(\chi)$ is irreducible. Harris and Taylor associate *two* n-dimensional representations of the Weil group to $I(\chi)$, one called $\mathrm{rec}_K(\pi)$ and the other called $r_l(\pi)$. Strictly speaking $\mathrm{rec}_K(\pi)$ is a complex representation and $r_l(\pi)$ is an l-adic one, but we fix an isomorphism $\overline{\mathbf{Q}}_l = \mathbf{C}$ so there's not much difference in this case. The point is that these representations are not "the same", they differ by a duality and a twist in general. I will explicitly write them both down here. I think that the point is that, vaguely speaking, $\mathrm{rec}_K(\pi)$ works better for things like matching L-functions but $r_l(\pi)$ works better for local-global compatibilities.

Let's fix notation as in the Harris-Taylor book. Let

$$\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$$

be the map sending ϖ, a uniformiser, to a geometric Frobenius. Let's fix a norm on W_K that's compatible with Art_K, that is, define $|g| = |\operatorname{Art}_K^{-1}(g)|$. So for example the norm of a geometric Frobenius is $1/q$.

Here is the explicit dictionary. If $\chi_i : K^\times \to \mathbf{C}^\times$ is a continuous character then let χ_i also denote the associated 1-dimensional representation of the Weil group. The dictionary in the 1-dimensional case is:

$$r_l(\chi) = \chi^{-1}$$

and

$$\mathrm{rec}_K(\chi) = \chi.$$

This follows from the remarks on p6 of the Harris-Taylor book. In particular, if χ is unramified and sends ϖ to α then $r_l(\chi)$ sends an arithmetic Frobenius to α and $\mathrm{rec}_K(\chi)$ sends a geometric Frobenius to α.

The general case:

$$r_l(I(\chi)) = \oplus_{i=1}^n \chi^{-1}.|.|^{(1-n)/2}$$

and

$$\mathrm{rec}_K(I(\chi)) = \chi$$

again; this also follows easily from remarks on p6 of Harris-Taylor.

7 A concrete example: $n = 2$ unramified principal series.

As a concrete example—if $n = 2$ and χ_1, χ_2 are unramified characters with $\chi_i(\varpi) = \alpha_i$, and $\chi_1/\chi_2 \neq |.|^{\pm 1}$, then $\pi := I(\chi)$ is unramified principal series, the usual Hecke operators T and S defined using double cosets have eigenvalues $\sqrt{q}(\alpha_1+\alpha_2)$ and $\alpha_1\alpha_2$ respectively, $r_l(\pi)$ sends a geometric Frobenius to a matrix with eigenvalues $\sqrt{q}/\alpha_1$ and $\sqrt{q}/\alpha_2$, and $\mathrm{rec}_K(\pi)$ sends a geometric Frobenius to a matrix with eigenvalues α_1, α_2.