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On the Langlands Correspondence for Algebraic Tori

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Abstract

We review some of the main results in class field theory, together with group homology and cohomology, in order to extend the results by R.P. Langlands in [1]. This is done by considering representations into any divisible abelian topological group. With this we can then prove what is known as the abelian case of the p-adic Langlands program.

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Introduction.

In this paper we will be looking one of the main themes in algebraic number theory, which is to understand the structure of field extensions by using Galois theory. Using class field theory, it is possible to describe all the extensions of a field (local or global) which have an abelian Galois group, by using the arithmetic of the field.

In the non-abelian case, class field theory no longer works and this is where the Langlands program takes over. In his paper [1], Langlands essentially translates class field theory, into the language of representation theory, which then allows us to generalize class field theory to non-abelian situations.

Langlands does this by relating representations of the Weil group of a finite Galois extension K of a field F into the L-group of an algebraic torus T, with representations of $T(\mathbb{A}_F)/T(F)$ into the multiplicative complex numbers, where $T(\mathbb{A}_F)$ ($T(F)$ resp.) are the $\mathbb{A}_F$-rational points (the F-rational points resp.) of T. The main goal of this paper is to extend these results by allowing representations of $T(\mathbb{A}_F)/T(F)$ into a more general class of groups. In particular, we want to look at representations into $\mathbb{C}_p^\times$ (the units of the completion of an algebraic closure of $\mathbb{Q}_p$), which gives us what is called the abelian case of the p-adic Langlands program.

In order even understand the theorems in Langlands' paper one must have a good working knowledge of group cohomology and group homology, together with a familiarity with the main results in class field theory as well as some knowledge of topological groups. With this in mind the first section is devoted to group cohomology and group homology. In this section we develop the theory in a very detailed way, since this will be our main tool later on. The second and third sections are in much less detail since we will not need many of the actual results in class field theory or about topological groups, but we do need to be aware of certain definitions and results.

1 Group Homology and Cohomology

1.1 G-modules

Let G be a group.

Definition 1.1. A G-**module** is an abelian group M on which G acts. This means that we have a group homomorphism $\phi : G \to \text{Aut}(M)$.

In practice we write $\phi(g)(m)$ simply as $g(m)$ or gm for $g \in G$ and $m \in M$. It then follows that in this notation, being a G-module means that for any $g, g' \in G$ and any $m, m' \in M$ we have:

(i) $g(m + m') = gm + gm'$;

(ii) $(gg')m = g(g'm)$.

Technically we have just defined a left G-module, but we can define a right G-module by letting G act on the right. From now on, in this section, by G-module we mean left G-module, unless otherwise stated.

Examples 1.2. (a) If L/K is a Galois field extension with Galois group $G = \text{Gal}(L/K)$, then by definition we have that L is a G-module, when we consider L as an additive group. Similarly if we look at the multiplicative group $L^\times$, this is also a G-module.

(b) For any abelian group A and any group G, we define the trivial action of G on A by $ga = a$ for all $a \in A$, $g \in G$, this then makes A into a G-module.

For two G-modules M and N we are interested in maps between them that preserve the G-module structure on M and N, which brings us to our next definition.

Definition 1.3. A **G-module homomorphism** (or a **G-linear map**) between two G-modules M and N is a group homomorphism $\phi : M \to N$ such that for all $g \in G$ and $m \in M$ we have

$$\phi(gm) = g\phi(m).$$

We denote the set of all G-module homomorphism from M to N by $\text{Hom}_G(M, N)$, and we let $\text{Hom}(M, N)$ denote abstract group homomorphisms.

Since we are working with abelian groups, we can extend the action of G on M to an action of $\mathbb{Z}[G]$ on M by

$$\left(\sum n_i g_i \right)(m) = \sum n_i(g_i(m)), \qquad \text{for any } n_i \in \mathbb{Z}, g \in G, m \in M,$$

where we define $n_i(g_i(m))$ to be

$$\overbrace{g_i m + \cdots + g_i m}^{n_i} \in M$$

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if $n_i > 0$, and similarly for $n_i < 0$ using $-g_i m$. We will use G-module and $\mathbb{Z}[G]$-module interchangeably from now on.

Now for a G-module M we will be interested in the set of all elements on M that are fixed by the action of G as this will play an important role later on.

Definition 1.4. If M is a G-module, define $M^G = \{m \in M \mid gm = m, \forall g \in G\}$

It can be checked easily that M^G is in fact a subgroup of M.

Example 1.5. If we look back at Example 1.2 we see that by Galois theory $L^G = K$ in (a) and $A^G = A$ in (b).

Proposition 1.6. *If A is a G-module, then $A^G \cong \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$, where we give $\mathbb{Z}$ the trivial $\mathbb{Z}[G]$-module action.*

Proof. Define
$$f : \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to A^G, \qquad \alpha \mapsto \alpha(1).$$

Then f is a group homomorphism, since clearly

$$f(\alpha + \beta) = \alpha(1) + \beta(1),$$

and it maps the identity to the identity. Also we have $\alpha(1) = \alpha(g.1) = g\alpha(1)$, where the first equality is because G acts trivially on $\mathbb{Z}$ and the second because α is G-linear, so we see that the image of f lies in A^G.

Now since $\mathbb{Z}$ is cyclic, any $\alpha \in \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$ is uniquely determined by its value on $1 \in \mathbb{Z}$, which gives us that f is injective, since
$$f(\alpha) = f(\beta) \implies \alpha(1) = \beta(1) \implies \alpha = \beta.$$

So we are left proving that f is surjective to complete the proof. This can be done by defining $\phi_a(n) = na$ for $n \in \mathbb{Z}$ and $a \in A^G$; then $\phi_a \in \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$ as G-linearity follows from the fact that G acts trivially on $\mathbb{Z}$ and that $a \in A^G$.

$\square$

In fact, it is clear that $\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) = A$.

1.2 Coinduced Modules

In this section we introduce the notion of a **coinduced** module for cohomology. These are G-modules that will turn out to have trivial cohomology groups (i.e. they are all zero), and other very useful properties that we will need later on.

Let H a subgroup of G and M an H-module, then we can make M into a G-module as follows:

Definition 1.7. Let G be a group and let H be a subgroup of G. Then for a H-module M we define

$$\operatorname{Coind}_H^G(M) = \{\varphi : G \to M \mid \varphi(hx) = h\varphi(x) \text{ for all } h \in H, x \in G\}.$$

Here the φ need not be homomorphisms. We can then make $\operatorname{Coind}_H^G(M)$ into a G-module by defining

$$(\alpha + \alpha')(x) = \alpha(x) + \alpha'(x);$$
$$(g\alpha)(x) = \alpha(xg),$$

where $\alpha, \alpha' \in \operatorname{Coind}_H^G(M)$ and $g, x \in G$. The first equation makes $\operatorname{Coind}_H^G(M)$ into an abelian group and the second gives it a G-action.

Proposition 1.8. *If M, N are H-modules and $\alpha : M \to N$ is an H-homomorphism, then α induces a map*

$$f^\alpha : \mathrm{Coind}_H^G(M) \to \mathrm{Coind}_H^G(N), \qquad f^\alpha(\varphi) = \alpha(\varphi),$$

such that

(a) If α is injective, then so is f^α.

(b) If α is surjective, then so is f^α.

Proof. (a) If α is injective, then from how f^α is defined, we see that f^α is also injective, since

$$f^\alpha(\varphi) = 0 \Leftrightarrow \alpha(\varphi) = 0 \Leftrightarrow \varphi = 0.$$

(b) We begin by first choosing a set of right coset representatives S for H in G. Then we can write

$$G = \bigcup_{s \in S} Hs.$$

Now since we are assuming that α is surjective, then for any $\varphi_m \in \mathrm{Coind}_H^G(M)$ and for each $s \in S$ we can find a $m_s \in M$ such that $\alpha(m_s) = \varphi_m(s) \in N$. This allows us to define $\varphi_m'(hs) = h \cdot m_s$, and we can see that with this definition $\varphi_m' \in \mathrm{Coind}_H^G(M)$.

If $g \in G$, then from above we see that $g \in Hs$ for some $s \in S$ so we can write $g = hs$ and we have

$$f^\alpha(\varphi_m')(g) = \alpha(\varphi_m'(hs)) = \alpha(h \cdot m_s) \overset{(*)}{=} h\alpha(m_s) = h\varphi_m(s) \overset{(**)}{=} \varphi_m(hs) = \varphi_m(g),$$

which shows that f^α is surjective (note (*) and (**) hold since α and φ_m are both H-linear).

$\square$

From this proposition we have:

Corollary 1.9. *If*

$$0 \longrightarrow A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0,$$

is an exact sequence of H-modules, then

$$0 \longrightarrow \mathrm{Coind}_H^G(A) \overset{f^\alpha}{\longrightarrow} \mathrm{Coind}_H^G(B) \overset{f^\beta}{\longrightarrow} \mathrm{Coind}_H^G(C) \longrightarrow 0$$

*is exact. In this case we say $\mathrm{Coind}_H^G(-)$ is an **exact covariant functor** (see definition 1.22).*

Proof. The fact that the sequence is exact at the first and last position, follows at once from the proposition above. So we only need to check exactness in the middle position. Now it is clear from the definition of f^α and f^β, that $\mathrm{Im}\, f^\alpha \subseteq \ker f^\beta$. So we need only check the reverse inclusion.

Assume that $f^\beta(\varphi) = 0$, for $\varphi \in \mathrm{Coind}_H^G(B)$. Then by the definition of f^β, we have $(\beta \circ \varphi)(g) = 0$ for all $g \in G$, which in turn means that $\varphi(g) = \alpha(a_g)$ for some $a_g \in A$. If we define $\psi(g) = a_g$, then it is easily checked that $\psi \in \mathrm{Coind}_H^G(A)$, and $f^\alpha(\psi) = \varphi$, hence the result.

$\square$

In the case $H = 1$, being an H-module is the same as being an abelian group, and in this case we write $\mathrm{Coind}^G(A)$ for $\mathrm{Coind}_H^G(A)$. We call a G-module M **coinduced** if there exists some abelian group A such that $M = \mathrm{Coind}^G(A)$. In fact we have

$$\mathrm{Coind}^G(A) = \mathrm{Hom}(\mathbb{Z}[G], A).$$

Now we prove a proposition showing how the original H-module and G-modules relate to the coinduced modules we make from them.

Proposition 1.10. *(a) Let M be a G-module and H any subgroup of G. Then there is an injective map*

$$\alpha : M \longrightarrow \mathrm{Coind}_H^G(M).$$

(b) Let H be a subgroup of G and let N be an H-module. Then we have a H-homomorphism

$$\psi : \mathrm{Coind}_H^G(N) \longrightarrow N, \qquad \varphi \mapsto \varphi(1).$$

(c) For any G-module M and for any H-module N, we have

$$\mathrm{Hom}_G(M, \mathrm{Coind}_H^G(N)) \cong \mathrm{Hom}_H(M, N),$$

where on the right hand side of this isomorphism we are thinking of M as a H-module.

Proof. (a) If for each $m \in M$ we set

$$\alpha(m) = \theta_m : x \mapsto xm, \qquad \text{for all } x \in G,$$

then θ_m is an H-homomorphism, since

$$\theta_m(hx) = hxm = h\theta_m(x), \qquad \text{for all } h \in H, x \in G,$$

hence $\theta_m \in \mathrm{Coind}_H^G(M)$. Furthermore if $\theta_m = \theta_n$, then we must have $xm = xn$ for all $x \in G$ from which it follows that $m = n$. So we see that α is in fact injective.

(b) Clearly we have $\varphi(1) \in N$, and for any $h \in H$, we have

$$\psi(h\varphi) = (h\varphi)(1) = h(\varphi(1)) = h\psi(\varphi).$$

Also we can see that ψ is a homomorphism since

$$\psi(\varphi + \varphi') = (\varphi + \varphi')(1) = \varphi(1) + \varphi'(1) = \psi(\varphi) + \psi(\varphi').$$

(c) Let ψ be as above, then define :

$$F : \mathrm{Hom}_G(M, \mathrm{Coind}_H^G(N)) \longrightarrow \mathrm{Hom}_H(M, N)$$

where for $m \in M$, $\beta \in \mathrm{Hom}_G(M, \mathrm{Coind}_H^G(N))$ we define $F(\beta))(m) = \psi(\beta(m))$. With this definition it is easy to check that F is a homomorphism and that $F(\beta)$ is H-linear.

Conversely let $\gamma \in \mathrm{Hom}_H(M, N)$, then define $F^{-1}(\gamma)$ to be the map:

$$F^{-1}(\gamma)(m)(g) = \gamma(gm), \qquad \text{for all } g \in G, m \in M.$$

It is easy to check that this map is a well-defined inverse for F.

$\square$

As an immediate consequence of 1.10 (c) we have that $\mathrm{Coind}_H^G(N)$ has the universal property that any H-homomorphism from a G-module into N factors uniquely through $\mathrm{Coind}_H^G(N)$.

For group homology we can define **induced** modules, which have similar properties to coinduced modules.

Definition 1.11. A G-module of the form $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$, for any abelian group A, is called an induced module[1].

In the case when G is a finite group the notion of induced and coinduced modules coincide, and more generally if G is any group and H is a subgroup of finite index, then one can show that $\mathrm{Coind}_H^G(A) \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A$ (see [2] p. 57 or [3] p. 769).

1.3   Cohomology Groups

In order to define the cohomology groups we first need some definitions.

Definition 1.12. A functor T from the category of G-modules (left or right) to the category of abelian groups is called an **exact covariant functor** if for any short exact sequence of G-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

the sequence

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \longrightarrow 0$$

is also a short exact sequence.

Similarly we say T is a **exact contravariant functor**, if it sends exact sequences to exact sequences, but if $f : A \longrightarrow B$, then

$$T(f) : T(B) \longrightarrow T(A).$$

If the functor T sends an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

to

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C),$$

then its called a **left exact functor**. As an example of this we have that $\mathrm{Hom}(X, -)$ is an left exact functor for any group X.

Definition 1.13. A G-module P is called **projective** if $\mathrm{Hom}_G(P, -)$ is an exact covariant functor, and is called **injective** if $\mathrm{Hom}_G(-, P)$ is an exact contravariant functor.

Definition 1.14. A **projective resolution** of a G-module M is an exact sequence

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where each P_i is a projective module. An **injective resolution** of M is an exact sequence

$$0 \longrightarrow M \longrightarrow R_0 \longrightarrow \cdots \longrightarrow R_n \longrightarrow R_{n+1} \longrightarrow \cdots$$

where each R_i is an injective module.

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[1]Here the action of G on $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ is given by $g \cdot (x \otimes a) = (g \cdot x) \otimes a$ for $g \in G$, $x \in \mathbb{Z}[G]$, and $a \in A$.

Proposition 1.15. *(a) A free $\mathbb{Z}[G]$-module is projective*

(b) For any group G, there exist a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules

Proof. (a) Using the fact that $\mathrm{Hom}(\mathbb{Z}[G], -)$ is a left exact functor, we can reduce the problem to checking that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of G-modules, then so is

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}[G], A) \longrightarrow \mathrm{Hom}(\mathbb{Z}[G], B) \longrightarrow \mathrm{Hom}(\mathbb{Z}[G], C).$$

So we only need to check that the last map is surjective. But this follows from the fact that generators of $\mathbb{Z}[G]$ uniquely define the homomorphisms from $\mathbb{Z}[G]$ to any G-module (see [3], p. 369).

(b) For any group G, let $P_n = \mathbb{Z}[G^{n+1}]$. Then P_n is $\mathbb{Z}$-free with a basis of $n+1$-tuples in G^{n+1} and has a G-action defined by

$$g(g_0, \ldots, g_n) = (gg_0, \ldots, gg_n).$$

We can then form the exact sequence

$$\cdots \xrightarrow{d} P_n \xrightarrow{d} P_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} P_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where

$$d(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n)$$

and

$$\epsilon : \sum_{i=1}^{n} n_i g_i \longmapsto \sum_{i=1}^{n} n_i. \qquad \textbf{(augmentation map).}$$

To show that this is in fact an exact sequence one can define a map $D : P_i \longrightarrow P_{i+1}$ such that $D(g_0, \ldots, g_i) = (1, g_0, \ldots, g_i)$, then using this one can prove that $dD + Dd = \mathbb{1}$ and that $dd = 0$ which gives exactness.

This projective resolution of $\mathbb{Z}$ is called the **standard resolution** or **bar resolution** of $\mathbb{Z}$.

$\qquad\quad\square$

Definition 1.16. Let A_i be a set of abelian groups for which we have the sequence of homomorphisms:

$$0 \longrightarrow A_0 \xrightarrow{d_1} A_1 \xrightarrow{d_2} \cdots \xrightarrow{d_n} A_n \xrightarrow{d_{n+1}} \cdots$$

(a) The sequence is called a **cochain complex** (or simply a complex) if $d_n \circ d_{n-1} = 0$ for all n.

(b) If we denote the above cochain complex by $A^\bullet$, then its nth **cohomology group** is the quotient $\ker(d_{n+1})/\mathrm{Im}(d_n)$, and we denote it by $H^n(A^\bullet)$.

If the arrows go in the other direction we call it a chain complex, and its nth **homology group** is $\ker(d_n)/\mathrm{Im}(d_{n+1})$ and is denoted by $H_n(A^\bullet)$.

Now let M, N be a R-modules for some ring R and let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_0 \xrightarrow{d_{-1}} M \longrightarrow 0$$

be any projective resolution for M, then we can form the complex

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \xrightarrow{d_{-1}'} \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1'} \mathrm{Hom}_R(P_1, N) \xrightarrow{d_2'} \cdots$$

where the d_i' maps are induced from d_i.

Now the cohomology groups for this complex are very useful and are called the nth **cohomology groups from the derived functor** $\mathrm{Hom}_R(-, N)$ and are denoted by

$$\mathrm{Ext}_R^n(M, N).$$

If $\mathbb{Z}$ and N are $\mathbb{Z}[G]$-modules for some group G (with G acting trivially on $\mathbb{Z}$), then using the standard resolution of $\mathbb{Z}$ we define

$$H^n(G, N) = \mathrm{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, N),$$

which we call the nth **cohomology group of G with coefficients in N**.

It is a fact that the cohomology groups are independent of the projective resolution used to compute them (see [3] p. 749). In practice it will be useful to have a more explicit definition of the $H^n(G, N)$ groups. To get this we reinterpret $\mathrm{Hom}_{\mathbb{Z}[G]}(P_i, N)$ by observing that its elements are determined uniquely by its values on the $\mathbb{Z}[G]$ basis of P_i. This leads us to the following definition.

Definition 1.17. Let A be a G-module and let G^0 to be the singleton set, then for $n \geq 0$, we define the (inhomogeneous) n-**cochains** of G with coefficients in A to be the set of all maps from G^n to A and we denote it by $C^n(G, A)$.

Note that with this definition we have $C^0(G, A) = A$ and that each $C^n(G, A)$ can be considered as an abelian group. Since for $n = 0$ we have the abelian group structure of A and for $n \geq 1$ we use the usual pointwise addition of functions.

Definition 1.18. We define a homomorphism

$$d_n : C^n(G, A) \longrightarrow C^{n+1}(G, A),$$

for $n \geq 0$, called the nth **coboundary homomorphism** as follows:

$$
\begin{aligned}
d_n(f)(g_1, \cdots, g_{n+1}) = {} & g_1 \cdot f(g_2, \cdots, g_{n+1}) \\
& + \sum_{i=1}^{n} (-1)^i (f(g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_{n+1})) \\
& + (-1)^{n+1} (f(g_1, \cdots, g_n)),
\end{aligned}
$$

where $f \in C^n(G, A)$.

It can be shown that not only is d_n is a homomorphism but that for $n \geq 1$ we have $d_n \circ d_{n-1} = 0$. Hence we have the following Corollary.

Corollary 1.19. *For a group G and a G-module A, we have that*

$$0 \longrightarrow C^0(G, A) \xrightarrow{d_0} C^1(G, A) \xrightarrow{d_1} \cdots \xrightarrow{d_n} C^{n+1}(G, A) \xrightarrow{d_{n+1}} \cdots$$

is a cochain complex.

Now we can use this complex to explicitly compute cohomology groups.

Definition 1.20. For G and A as in Corollary 1.19, we define:

- Let $Z^n(G, A) = \ker(d_n)$ for $n \geq 0$. We call its elements n-**cocycles**.

- Let $B^n(G, A) = \text{Im}(d_{n-1})$ for $n \geq 0$. We call its elements the n-**coboundaries**.

From this and the definition of the cohomology groups we have that $H^n(G, A) = Z^n(G, A)/B^n(G, A)$. We will now use this to compute some examples.

Examples 1.21. Let A be a G-module.

1. Recall that $C^0(G, A) = A$. Then $Z^0(G, A) = \ker(d_0) = A^G$, since for $f \in C^0(G, A)$ we have

$$d_0(f)(g) = gf - f.$$

So if $f \in \ker(d_0)$, then we have $gf = f$ for all $g \in G$. Hence $\ker(d_0) = A^G$ and since $B^0(G, A) = \text{Im}(d_{-1}) = 0$ we have

$$H^0(G, A) = A^G.$$

2. If $f \in C^1(G, A)$, then from the definition of d_1 we get

$$d_1(f)(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1).$$

Moreover, if $f \in Z^1(G, A)$, then we have that $d_1(f) = 0$, and thus

$$f(g_1 g_2) = g_1 \cdot f(g_2) + f(g_1), \qquad \text{for all } g_1, g_2 \in G.$$

Such an f is called a **crossed homomorphism**. Also note that from example (1) we have that

$$B^1(G, A) = \text{Im}(d_0) = \{\varphi \in C^1(G, A) \mid \varphi(g) = ga - a, \text{ for } a \in A, g \in G\},$$

which are also crossed homomorphisms, but these are called **principal crossed homomorphisms**. Then

$$H^1(G, A) = \frac{\{\text{all crossed homomorphisms from } G \longrightarrow A\}}{\{\text{ all the principal crossed homomorphisms}\}}.$$

Observe that if G acts trivially on A, then we see that $ga = a$ for all $a \in A$ and $g \in G$, so in this case $B^1(G, A) = 0$ and $Z^1(G, A) = \text{Hom}(G, A)$. Hence $H^1(G, A) = \text{Hom}(G, A)$.

Proposition 1.22. *Let G be the trivial group $G = 1$, and let A be a G-module (clearly with the trivial action). Then*

$$H^0(G, A) = A,$$
$$H^n(G, A) = 0, \qquad \text{for all } n \geq 1.$$

Proof. For $n = 0$, this follows from Example 1.21 (1) and the fact that G acts trivially on A. Now fix n, then each $f \in C^n(G, A)$ is determined by $f(1, \ldots, 1)$. So if for all $a \in A$ we define f_a to be such that $f_a(1, \ldots, 1) = a$, then we can identify $C^n(G, A) = A$ for all $n \geq 0$. Furthermore note that

$$d_n(f_a)(1, \ldots, 1, 1) = a + \sum_{i=1}^{n} (-1)^i a + (-1)^{n+1} a.$$

From which we get that $d_n = 0$ if n is even and d_n is the identity map when n is odd, and from this the result follows at once. $\square$

Lemma 1.23. *Let G be a finite cyclic group of order n with generator σ, and let A be a G-module. Then*

$$H^n(G, A) = \begin{cases} A^G & \text{if } n = 0 \\ A^G/NA & \text{if } n \text{ is even}, n \geq 2 \\ {_N}A/I_G A & \text{if } n \text{ is odd}, n \geq 1 \end{cases}$$

where

1. $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1}$ *and NA is the subgroup it generates in A,*

2. ${_N}A = \{a \in A \mid N(a) = 0\} = \ker(N),$

3. $I_G A$ *is the subgroup of A which is generated by the set $\{\sigma a - a \mid a \in A\}$.*

Proof. Note that since G is cyclic, $\mathbb{Z}[G]$ is a free group, and therefore by Proposition 1.15 (a) we have that it is projective. Also note that we have

$$N \circ (\sigma - 1) = (\sigma - 1) \circ N = \sigma^n - 1 = 0,$$

which when used with the fact that G is finite and cyclic gives

$$\operatorname{Im} N = N(\mathbb{Z}[G]) = \mathbb{Z}[G]^G = \ker((\sigma - 1));$$
$$\operatorname{Im}((\sigma - 1)) = (\sigma - 1)\mathbb{Z}[G] = \ker(N).$$

We can use this to define a projective resolution of $\mathbb{Z}$ as follows:

$$\xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma - 1} \mathbb{Z}[G] \xrightarrow{N} \cdots \xrightarrow{\sigma - 1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where ϵ is the augmentation map. Then by applying $\operatorname{Hom}(-, A)$ to this exact sequence we get the complex

$$\cdots \xleftarrow{N} A \xleftarrow{\sigma - 1} A \xleftarrow{N} A \xleftarrow{\sigma - 1} \cdots$$

after using the identification $\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) = A$. The result then follows at once.

$\square$

The last example we are going to look at is one that is encountered many times in class field theory, and the result is know as "Hilbert's Theorem 90".

Proposition 1.24. (Hilbert's Theorem 90) *Let L be a finite Galois extension of a field K, and let $G = \operatorname{Gal}(L/K)$. Then*

$$H^1(G, L^\times) = 0.$$

Proof. (Based on [2] Propositon 1.22, p. 65.) The result will follow if we can show that any 1-cocycle is in fact a 1-coboundary. If we use multiplicative notation for the 1-cocycles, we see that, by definition, they are homomorphisms from G to $L^\times$ such that

$$\alpha(xy) = x\alpha(y) \cdot \alpha(x) \qquad \text{for all } x, y \in G.$$

In order to show that α is a 1-coboundary, we need to find $\lambda \in L^\times$ such that $\alpha(x) = x(\lambda)/\lambda$ for all $x \in G$. With this in mind, take any $s \in L^\times$ and let

$$b = \sum_{x \in G} \alpha(x) \cdot xs.$$

Then there will exist some s such that $b \neq 0$, since by a standard result, we have that a finite set of distinct homomorphisms from any group G to $L^\times$ will be linearly independent. Therefore we can assume $b \neq 0$. Now take $g \in G$ and look at $g(b)$, by definition this will be

$$g(b) = \sum_{x \in G} g\alpha(x) \cdot gxs = \sum_{x \in G} \alpha(g)^{-1}\alpha(gx)gxs = \alpha(g)^{-1}b,$$

and this tells us that in fact α is a 1-coboundary, so we are done.

$\square$

Next we look at how we can use the cohomology groups make the cohomological extension of the functor $(-)^G$.

Proposition 1.25. *If*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of G-modules, then

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

is also a short exact sequence of groups. Hence $(-)^G$ is a **left exact functor.**

Proof. Now it is a basic result that $\operatorname{Hom}(A, -)$ is a left exact functor, and it is easy to show from the definitions that

$$\operatorname{Hom}_G(A, B) = (\operatorname{Hom}(A, B))^G.$$

It follows by 1.6 that $(-)^G$ is a left exact functor, hence the result. $\square$

Next we want to use the cohomology groups to make what is called a **cohomological extension** of the functor $(-)^G$

Definition 1.26. The cohomological extension of the functor $(-)^G$ is the long exact sequence:

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta_0} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \xrightarrow{\delta_1} H^2(G, A) \longrightarrow \cdots$$

where the δ_i are called **connecting** homomorphisms.

The proof that such a sequence exists and is the only one up to canonical equivalence can be found in [3] p. 751 or [4] p. 95, along with the definition of the connecting homomorphisms.

Now we say a G-module T has trivial cohomology groups if $H^n(G, T) = 0$ for all $n > 0$.

Lemma 1.27. (Dimension Shifting) *Assume we have a exact sequence of G-modules*

$$0 \longrightarrow M \longrightarrow T \longrightarrow P \longrightarrow 0$$

and that T is has trivial cohomology groups. Then when we compute the long exact sequence from 1.26 we get

$$0 \longrightarrow M^G \longrightarrow T^G \longrightarrow P^G \xrightarrow{\delta_0} H^1(G, M) \longrightarrow 0$$

and

$$H^n(G, M) \cong H^{n-1}(G, P), \qquad \text{for all } n \geq 1.$$

Proof. If we write out the long exact sequence from definition 1.26 for this particular short exact sequence we get

$$0 \longrightarrow M^G \longrightarrow T^G \longrightarrow P^G \xrightarrow{\delta_0} H^1(G, M) \longrightarrow H^1(G, T) \longrightarrow H^1(G, P) \xrightarrow{\delta_1} H^2(G, M) \longrightarrow H^2(G, T) \longrightarrow \cdots$$

Now for $n > 0$ replace the $H^n(G, T)$ with zero. Then the result follows by noting that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow 0$$

is exact, then $A \cong B$.

$\square$

Now this Lemma does not tell us if such a module with trivial cohomology even exists, but we can construct it by using Proposition 1.10 (a) with $H = 1$. This tells us that every G-module can be injected in to a coinduced module. So if we can prove that all coinduced modules have trivial cohomology groups, we will then be able to always use dimension shifting for any G-module.

Lemma 1.28. (Shapiro's Lemma) *Let H be a subgroup of G, then for any G-modules M and any $n \geq 0$:*

$$H^n(G, \operatorname{Coind}_H^G(M)) \cong H^n(H, M)$$

Proof. We begin by using the standard projective resolution of $\mathbb{Z}$ as in Proposition 1.15 (c). Since we want to compute $H^n(G, \operatorname{Coind}_H^G(M))$, we look at the complex formed by the groups

$$\operatorname{Hom}_{\mathbb{Z}[G]}(P_n, \operatorname{Coind}_H^G(M)).$$

Then using Proposition 1.10 we get that as complexes

$$\operatorname{Hom}_{\mathbb{Z}[G]}(P^\bullet, \operatorname{Coind}_H^G(M)) \cong \operatorname{Hom}_{\mathbb{Z}[H]}(P^\bullet, M),$$

and since isomorphic cochain complexes have isomorphic cohomology groups, the result follows.

$\square$

1.4   Change of Groups

Now that we have a working definition of the cohomology groups we will now look at how maps between groups and maps between G-modules affect the the cohomology groups. For this we first need the following definition.

Definition 1.29. Let G and G' be two groups, and let M be a G-module and M' a G'-module. Then the we say the homomorphisms

$$\varphi : G' \longrightarrow G \qquad \text{and} \qquad \psi : A \longrightarrow A'$$

are **compatible** if for all $g' \in G'$ we have

$$\psi(\varphi(g')a) = g'\psi(a).$$

The usefulness of these compatible homomorphisms is that they will induce maps between $C^n(G', A')$ and $C^n(G, A)$ and hence we will get induced maps between the corresponding cohomology groups.
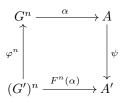
Proposition 1.30. *If the homomorphisms:*

$$\varphi : G' \longrightarrow G \qquad and \qquad \psi : A \longrightarrow A',$$

are compatible, then they induce a homomorphism

$$H^n(G, A) \longrightarrow H^n(G', A').$$

Proof. We begin by noting that we can use φ to induce a homomorphism $\varphi^n : (G')^n \longrightarrow G^n$. This leads us to define F^n such that for all $\alpha \in C^n(G, A)$, the following diagram commutes:

$$
\begin{array}{ccc}
G^n & \xrightarrow{\ \alpha\ } & A \\
\varphi^n \uparrow & & \downarrow \psi \\
(G')^n & \xrightarrow{\ F^n(\alpha)\ } & A'
\end{array}
$$

Then for any $\alpha \in C^n(G, A)$, we have:

$$F^n(\alpha) = \psi \circ \alpha \circ \varphi^n.$$

This gives us a homomorphism between the groups of cochains, but we need to show that in fact they give rise to a homomorphism between the cohomology groups. To do this we look at the commutative diagram:

$$
\begin{array}{ccccc}
\cdots \xrightarrow{\ d_{n-1}\ } & C^n(G, A) & \xrightarrow{\ d_n\ } & C^{n+1}(G, A) & \xrightarrow{\ d_{n+1}\ } \cdots \\
& \downarrow F^n & \text{S(n)} & \downarrow F^{n+1} & \\
\cdots \xrightarrow{\ d'_{n-1}\ } & C^n(G', A') & \xrightarrow{\ d'_n\ } & C^{n+1}(G', A') & \xrightarrow{\ d'_{n+1}\ } \cdots
\end{array}
$$

Then we can use the fact that φ and ψ are compatible homomorphisms to see that each square S(n) is a commutative square, i.e.,

$$F^{n+1} \circ d_n = d'_n \circ F^n,$$

from which it follows that F^n induces a homomorphism between the nth cohomology groups.

$\square$

This leads us to define three special maps between cohomology groups.

Corollary 1.31. *Let G be a group and A a G-module.*

(a) Let H is a subgroup of G and take $G' = H$ in 1.30. Then we get two compatible homomorphisms

$$\varphi : H \longrightarrow G \qquad and \qquad \mathbb{1} : A \longrightarrow A,$$

where φ is just the natural embedding of H into G, and $\mathbb{1}$ is the identity map. These are compatible homomorphisms and they give us the **restriction** *homomorphisms:*

$$\mathrm{Res} : H^n(G, A) \longrightarrow H^n(H, A).$$

It follows that if f is an n-cocycle, then it is a map $f : G^n \longrightarrow A$, and thus we have that

$$\mathrm{Res}(f) = f|_H : H^n \longrightarrow A.$$

Hence the name restriction.

(b) *Let H is a normal subgroup of G. We can naturally view A^H as a G/H-module. Then take $\varphi : G \longrightarrow G/H$ to be usual quotient map, and $\psi : A^H \longrightarrow A$ to be the natural inclusion. Once again these are compatible maps and they give us the* **inflation** *homomorphisms:*

$$\mathrm{Inf} : H^n(G/H, A^H) \longrightarrow H^n(G, A)$$

If we work through this definition we see that $\mathrm{Inf}(f)(g) = f(\overline{g})$ where $\overline{g}$ is the image of $g \in G$ in G/H

(c) *Let H is a subgroup of G of finite index m and let S be a set of representatives for the left cosets of H in G. We define a map*

$$\alpha \mapsto \sum_{s \in S} s\alpha(s^{-1}) : \mathrm{Coind}_H^G(A) \longrightarrow A$$

It can be verified that this definition is independent of choice of coset representatives and that it is in fact G-linear. If we denote this map by ψ, then letting φ in Proposition 1.30 be the identity map between from G to G, we get that ψ being compatible with φ is the same as ψ being G-linear. Hence we get a group homomorphism

$$H^n(G, \mathrm{Coind}_H^G(A)) \longrightarrow H^n(G, A).$$

Now recalling the result from Shapiro's Lemma and observing that A is also an H-module we have

$$H^n(G, \mathrm{Coind}_H^G(A)) \cong H^n(H, A).$$

The composition of these two homomorphisms gives us what is called the **corestriction** *homomorphism:*

$$\mathrm{Cor} : H^n(H, A) \longrightarrow H^n(G, A).$$

Proposition 1.32. *Let H be a subgroup of G of finite index m and let A be a G-module. Then the map*

$$\mathrm{Cor} \circ \mathrm{Res} : H^n(G, A) \longrightarrow H^n(G, A)$$

is multiplication by m for all $n \geq 0$.

Proof. We obtain the map $\mathrm{Cor} \circ \mathrm{Res}$ by looking at composition of the maps

$$A \longrightarrow \mathrm{Coind}_H^G(A) \longrightarrow A.$$

Now for the first map we can use the map θ_a defined in Proposition 1.10 (a) and for the second map we use the one from Corollary 1.31 (c), where we choose S to be a set of coset representatives for the left cosets of H. On composing them we get

$$a \mapsto \theta_a \mapsto \sum_{s \in S} s\theta_a(s^{-1}) = \sum_{s \in S} s(s^{-1}a) = \sum_{s \in S} a = ma.$$

$\square$

From this proposition it follows at once that if G is a finite group of order m and A is a G-module, then taking $H = 1$ in the proposition above, gives us that for all $n \geq 1$ we have $mH^n(G, A) = 0$.

We next prove a very useful connection between the inflation and restriction homomorphisms.

Lemma 1.33. (The Inflation-Restriction Exact Sequence) *Let H be a normal subgroup of G, and let A be a G-module then:*

(a) The sequence

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(H, A)$$

is exact.

(b) If $H^i(H, A) = 0$ for $0 < i < n$, then the sequence

$$0 \longrightarrow H^n(G/H, A^H) \xrightarrow{\text{Inf}} H^n(G, A) \xrightarrow{\text{Res}} H^n(H, A)$$

is exact.

Proof. (a)

- **Inf is injective:** In order to show it is injective we want to show that $\ker(\text{Inf}) = 0$. First take a moment to think about what the zeros are in $H^1(M, N)$. These are the 1-coboundaries. So if we look at how the inflation map works, it takes 1-cocycles $f : G/H \longrightarrow A^H$ and gives us a map $\text{Inf}(f) = \overline{f}$ such that

$$\overline{f} : G \longrightarrow G/H \longrightarrow A^H \longrightarrow A$$

  which is constant an the cosets of H. So if $\overline{f} \in \ker(\text{Inf})$, then $\overline{f}$ is a 1-coboundary, which means that there exists an $a \in A$ such that $\overline{f}(g) = ga - a$ for all $g \in G$. But since $\overline{f}$ is constant on cosets of H, then for any $h \in H$ we have $\overline{f}(gh) = \overline{f}(g)$, which gives us $ha = a$ for all $h \in H$. Hence we have that $f(g) = ga - a$ for all $g \in G$, so it is in fact a 1-coboundary as well. Therefore Inf is injective.

- **Exactness at $H^1(G, A)$:** Let $f : G \longrightarrow A$ be a 1-cocycle such that $\text{Res}(f)$ is a 1-coboundary, in other words for some $a \in A$ we have $f(h) = ha - a$ for all $h \in H$. Now since f is defined up to a coboundary we can define

$$f'(g) = f(g) - (ga - a),$$

  and f' will represent the same class in $H^1(G, A)$ as f, but $f'(h) = 0$ for any $h \in H$, so f' is constant on cosets of H. Hence it must come from the inflation of a 1-cocycle in $H^1(G/H, A^H)$, showing we have exactness at $H^1(G, A)$.

(b) Now assume that $H^i(H, A) = 0$ for $0 < i < n$. Then for $n = 1$ we know the claim is true by part (a). So assume that $n > 1$ and that the sequence is exact for $n - 1$. Then we are going to use dimension shifting to prove the result. Recall we showed in Proposition 1.10 (a) that we can inject A into $\text{Coind}^G(A)$, so we can then form the exact sequence

$$0 \longrightarrow A \longrightarrow \text{Coind}^G(A) \longrightarrow \text{Coind}^G(A)/A \longrightarrow 0.$$

Then since $\text{Coind}^G(A)$ is cohomologically trivial we can use dimension shifting to see that

$$H^i(H, \text{Coind}^G(A)/A) \cong H^{i+1}(H, A) \qquad \text{for } i > 0.$$

Now let us denote $\operatorname{Coind}^G(A)/A$ by A'. It then follows by induction that

$$0 \longrightarrow H^{n-1}(G/H, (A')^H) \xrightarrow{\text{Inf}} H^{n-1}(G, A') \xrightarrow{\text{Res}} H^{n-1}(H, A')$$

is exact, and then by dimension shifting we have that

$$0 \longrightarrow H^n(G/H, A^H) \xrightarrow{\text{Inf}} H^n(G, A) \xrightarrow{\text{Res}} H^n(H, A)$$

is also exact.

$\square$

Remark 1.34. *By using the Lyndon–Hochschild–Serre spectral sequence, the exact sequence above can be extended to the following exact sequence*

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(H, A)^{G/H} \xrightarrow{d} H^2(G/H, A^H) \xrightarrow{\text{inf}} H^2(G, A).$$

(See [5], p. 196).

1.5 Group Homology

Recall we defined for a G-module A, the subgroup A^G on which G acts trivially. It is easy to see that actually this is the largest subgroup of A on which G acts trivially. We then showed that the functor $(-)^G$ could be cohomologically extended using the cohomology groups. One can do a similar procedure but using the largest quotient A_G of A on which G acts trivially. This can be written as $A_G = A/I_G$ where I_G is called the **augmentation ideal** and is the ideal in $\mathbb{Z}[G]$ generated by all $g-1$ for $g \in G$. Then we can define a homological extension of the functor $(-)_G$ to get the homology groups $H_n(G, A)$, for $n \geq 0$.

So we begin as before, with a group G (finite or infinite) and the standard projective resolution $\mathbb{Z} \leftarrow P_\bullet$ of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules. Now let A be a G-module, if we apply the functor $(- \otimes_{\mathbb{Z}[G]} A)$ to this projective resolution, then we get a complex

$$0 \longleftarrow P_0 \otimes_{\mathbb{Z}[G]} A \xleftarrow{d_1} P_1 \otimes_{\mathbb{Z}[G]} A \xleftarrow{d_2} \cdots,$$

and the homology groups of this sequence are denoted by $H_n(G, A)$. As before we can write

$$Z_n(G, A) = \ker d_n, \qquad B_n(G, A) = \operatorname{Im} d_{n+1}, \qquad H_n(G, A) = \frac{Z_n(G, A)}{B_n(G, A)},$$

where the elements in $Z_n(G, A), B_n(G, A)$ are called n-cycles and n-boundaries, respectively.

If instead we had taken a projective resolution of a G-module M, and done the same as above, the homology groups would be denoted $\operatorname{Tor}_n^G(M, N)$, which are know as **torsion** groups.

It is not hard to show that for any group G and G-module A, that $A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$, which is why we can think of homology groups are homological extension of the functor $(-)_G$. Now with this we can take any short exact sequence of G-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

and form the complex

$$\cdots \xrightarrow{\delta_{n+1}} H_n(G, A) \longrightarrow H_n(G, B) \longrightarrow H_n(G, C) \xrightarrow{\delta_n} H_{n-1}(G, A) \longrightarrow \cdots \xrightarrow{\delta_0} A_G \longrightarrow B_G \longrightarrow C_G \longrightarrow 0,$$

where the δ_i are boundary maps and we have $H_0(G, A) = A_G$ (see [4] p. 97, or [2] p. 72).

In what follows it will be useful to show how the elements of $H_1(G, A)$ are defined since we will be working extensively with them in the last section.

Definition 1.35. Let G be any group and A a G-module. Then $H_1(G, A) = Z_1(G, A)/B_1(G, A)$, so its elements can be viewed 1-cycles modulo 1-boundaries. Now we can think of $x \in Z_1(G, A)$ as a map $x : G \to A$ such that $x(g) = 0$ for all but finitely many $g \in G$, and such that

$$\sum_{g \in G} (g^{-1} - 1)x(g) = 0.$$

Also note that if y is a 2-chain, then $\delta_2 y \in B_1(G, A)$ is defined as

$$(\delta_2 y)(g) = \sum_{h \in G} h^{-1} y(h, g) - \sum_{h \in G} y(gh^{-1}, h) + \sum_{h \in G} y(g, h).$$

Now for the exact sequence of G-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

we can (following [4], p. 98) give an explicit description of the connecting homomorphism

$$\delta_1 : H_1(G, C) \longrightarrow H_0(G, A)$$

as follows. Take a 1-cycle $x \in H_1(G, A)$, then let

$$dx = \sum_{g \in G} (g^{-1} - 1)x(g),$$

then since x is a 1-cycle, we must have $dx = 0$. Now for each $g \in G$, we lift $x(g)$ to $\bar{x}(g)$ in B. Then we have $d\bar{x} = 0$, so by exactness it must be an element of A. Then we can define $\delta_1(x)$ to be the class of $d\bar{x}$ in A.

We can also define maps between homology groups as we did for cohomology groups. Most of the maps we have already defined have an analogue for group homology. For example the corestriction map defined before, has an analogue for homology groups called the **transfer map** (or **restriction**), inflation has an analogue called coinflation and restriction has an analogue called corestriction, and we have an analogue for the Lyndon–Hochschild–Serre spectral sequence, which, for any group G, H a normal subgroup of G and A a G-module; gives an exact sequence

$$H_2(G, A) \xoverset{\text{Coinf}}{\longrightarrow} H_2(G/H, A_H) \xoverset{d}{\longrightarrow} H_1(H, A)_{G/H} \xoverset{\text{Cor}}{\longrightarrow} H_1(G, A) \xoverset{\text{Coinf}}{\longrightarrow} H_1(G/H, A_H) \longrightarrow 0.$$

(See [5], p. 196).

Later we will be working extensively with the transfer map, so we note here some of its defining properties.

Proposition 1.36. *Let G be any group, and let H be a subgroup of G of finite index with $\{g_i\}$ denoting left coset representatives of H in G. Then for $n \geq 0$ and any G-module A, there exists unique homomorphisms*

$$\mathrm{Tr}_n : H_n(G, A) \longrightarrow H_n(H, A),$$

such that

1. *For $n = 0$, and all $a \in A$, we have*

$$\mathrm{Tr}_0(\bar{a}) = \sum_i \overline{g_i a},$$

17

where on the left $\bar{a}$ denotes the image of a in A_G and on the right $\overline{g_i a}$ denotes the image of $g_i a$ in A_H.

2. If $0 \xrightarrow{i} A \longrightarrow B \xrightarrow{p} C \longrightarrow 0$ is an exact sequence of G-modules, then there is a commutative diagram

$$
\begin{array}{ccccccc}
H_n(G,B) & \xrightarrow{p'} & H_n(G,C) & \xrightarrow{\delta} & H_{n-1}(G,A) & \xrightarrow{i'} & H_{n-1}(G,B) \\
\downarrow{\scriptstyle \mathrm{Tr}_n} & & \downarrow{\scriptstyle \mathrm{Tr}_n} & & \downarrow{\scriptstyle \mathrm{Tr}_{n-1}} & & \downarrow{\scriptstyle \mathrm{Tr}_{n-1}} \\
H_n(H,B) & \xrightarrow{p'} & H_n(H,C) & \xrightarrow{\delta} & H_{n-1}(H,A) & \xrightarrow{i'} & H_{n-1}(H,B)
\end{array}
$$

Proof. See [6] Proposition 9.93, p. 576. $\qquad\square$

1.6 Tate Cohomology Groups

In order to define the Tate cohomology groups we need to restrict our attention to finite groups G. Under this assumption we can, in a sense, "join up", the long exact sequence for homology and cohomology. This involves extending the standard resolution of $\mathbb{Z}$, to a doubly-infinite complete resolution (one which extends infinitely in both directions) and then we will be able to define a complete resolution of any G-module. From this we can then define the Tate cohomology groups as the cohomology groups of this new complete resolution.

From now on in this section all groups G will be finite.

Definition 1.37. Let G be a finite group and A be a G-module. We define the **Norm map** N_G to be the map

$$
N_G : A \longrightarrow A, \qquad a \mapsto \sum_{g \in G} ga.
$$

Definition 1.38. Let A be a G-module. The **norm residue group** or **zeroth Tate cohomology group** is defined to be

$$
\widehat{H}^0(G,A) = A^G / N_G(A).
$$

Proposition 1.39. *For any G-module A we have*

$$
\mathrm{Im}(N_G) \subseteq A^G, \qquad and \qquad I_G A \subseteq \ker(N_G),
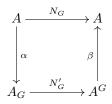$$

where we define $I_G A$ as in 1.23

Proof. Let $x \in G$ and $a \in A$, then we have that

$$
N_G(xa) = \sum_{g \in G} gxa = \sum_{gx \in G} a = N_G(a) \Rightarrow I_G A \subseteq \ker N_G.
$$

Similarly we have

$$
x N_G(a) = N_G(a) \Rightarrow \mathrm{Im}(N_G) \subseteq A^G.
$$

$\qquad\square$

Corollary 1.40. *The map N_G induces a map N'_G between A_G and A^G, such that*

$$
\begin{array}{ccc}
A & \xrightarrow{\ N_G\ } & A \\
\downarrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\
A_G & \xrightarrow{\ N'_G\ } & A^G
\end{array}
$$

is a commutative diagram, where α, β are the quotient and inclusion homomorphisms respectively.

With this we can construct the following exact sequence

$$
0 \longrightarrow \ker(N_G)/I_G A \longrightarrow A_G \xrightarrow{\ N'_G\ } A^G \longrightarrow \widehat{H}^0(G, A) \longrightarrow 0.
$$

It is common practice to abuse notation here and denote N'_G simply by N_G.

Definition 1.41. For any G-module A we define

$$
\widehat{H}^{-1}(G, A) = \ker(N_G)/I_G A
$$

The two groups $\widehat{H}^0, \widehat{H}^{-1}$ are examples of what we will come to call Tate cohomology groups. Next we will construct all the Tate cohomology groups $\widehat{H}^n(G, A)$ for $n \in \mathbb{Z}$. It will then be easy to see that for $n \geq 1$ we have $\widehat{H}^n(G, A) = H^n(G, A)$, and that for $n < -1$ we have $\widehat{H}^n(G, A) = H_n(G, A)$.

A **complete resolution** of $\mathbb{Z}$ can be made by using the standard projective resolution we worked with before and for each P_i we define its dual to be $P_i^* = \operatorname{Hom}(P_i, \mathbb{Z})$, which satisfies the following

Proposition 1.42. *Let A be a G-module.*

1. *$A^* = \operatorname{Hom}(A, \mathbb{Z})$ is a G-module with the G-action defined by*

$$
(ga^*)(x) = a^*(g^{-1}x)
$$

for $g \in G, x \in A, a^* \in A^*$.

2. *If A is G-free with a finite basis over $\mathbb{Z}[G]$, then so is A^*.*

Proof. see [7] p. 20. $\qquad\qquad\square$

We now have two $\mathbb{Z}[G]$-resolutions of $\mathbb{Z}$

$$
\cdots \xrightarrow{\ d\ } P_1 \xrightarrow{\ d\ } P_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \ .
$$

$$
0 \longrightarrow \mathbb{Z} \longrightarrow P_0^* \xrightarrow{\ d'\ } P_1^* \xrightarrow{\ d'\ } \cdots
$$

Here d is defined as in 1.18 and

$$
d'(g_0^*, \ldots, g_{n-1}^*) = \sum_{\tau \in G} \sum_{i=0}^{n-1} (-1)^i (g_0^*, \ldots, g_{i-1}^*, \tau^*, g_i^*, \ldots, g_{n-1}^*)
$$

where the (g^*) represent the $\mathbb{Z}$-basis of P_i^*. The second resolution is exact since each P_i is a finitely generated $\mathbb{Z}$-free G-module. If we then splice these two resolutions together and rename $P_{i-1}^* = P_{-i}$ we will get the complete resolution of $\mathbb{Z}$

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} P_{-1} \xrightarrow{\partial_{-1}} P_{-2} \xrightarrow{\partial_{-2}} \cdots$$

with

$$\partial_n = \begin{cases} d & \text{if } n \geq 1 \\ g_0 \mapsto \sum_{s \in G}(s^*) & \text{if } n = 0 \\ d' & \text{if } n < 0. \end{cases}$$

Definition 1.43. For a G-module A, the **complete standard resolution of** A is the sequence

$$\cdots \xrightarrow{\delta^{-2}} X^{-2} \xrightarrow{\delta^{-1}} X^{-1} \xrightarrow{\delta^0} X^0 \xrightarrow{\delta^1} X^1 \xrightarrow{\delta^2} X^2 \xrightarrow{\delta^3} \cdots$$

where $X^i = X^i(G, A) = \operatorname{Hom}(P_i, A)$ for $i \in \mathbb{Z}$ and $\delta^i(f) = f \circ \partial_i$. In practice we will drop the subscripts and superscripts and just denote denote them as δ, ∂.

Proposition 1.44. *For a G-module A, the complete standard resolution is exact.*

Proof. (Sketch) The idea is to construct a
$$D : X^{i+1} \longrightarrow X^i$$

such that $\delta D + D\delta = \mathbb{1}$ and $\delta\delta = 0$. For details see [8] p. 23. $\qquad \square$

If we then apply the functor $(-)^G$ to this sequence we get the complex $\widehat{C}^\bullet(G, A) = ((X^n)^G)_{n \in \mathbb{Z}}$ (with the notation as above), then the cohomology groups of this complex are called the **Tate cohomology groups** and are denoted $\widehat{H}^n(G, A)$ for $n \in \mathbb{Z}$. In order to see that for $n < -1$ we have $\widehat{H}^n(G, A) = H_n(G, A)$, we have the following proposition.

Lemma 1.45. *Let C be a finitely generated $\mathbb{Z}$-free G-module*

(a) We have for any G-module A
$$C \otimes A \cong \operatorname{Hom}(C^*, A).$$

(b) We have also
$$C \otimes_{\mathbb{Z}[G]} A \cong \operatorname{Hom}_G(C^*, A)$$

Proof. For part (a) see proof of Proposition 4.5. For part (b), see [4] p. 103. $\qquad \square$

We can now recover the usual definition of the Tate cohomology groups as:

$$\widehat{H}^n(G, A) = \begin{cases} H^n(G, A) & \text{if } n > 0 \\ A^G/N_G(A) & \text{if } n = 0 \\ \ker(N_G)/I_G A & \text{if } n = -1 \\ H_{-n}(G, A) & \text{if } n < -1. \end{cases}$$

With this description we get a more explicit definition of the Tate cohomology groups, that allows us to easily prove the following theorem.

Theorem 1.46. *For any short exact sequence of G-modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we can form the exact sequence

$$\cdots \longrightarrow \widehat{H}^n(G,A) \longrightarrow \widehat{H}^n(G,B) \longrightarrow \widehat{H}^n(G,C) \xrightarrow{\delta} \widehat{H}^{n+1}(G,A) \longrightarrow \cdots$$

which extends infinitely in both directions.

Proof. From Proposition 1.39, we see that we can use the N_G map to connect the zeroth homology and cohomology groups, so we can form the diagram

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H_1(G,C) & \longrightarrow & H_0(G,A) & \longrightarrow & H_0(G,B) & \longrightarrow & H_0(G,C) & \longrightarrow 0 \\
 & & & \downarrow{\scriptstyle N_G} & & \downarrow{\scriptstyle N_G} & & \downarrow{\scriptstyle N_G} & \\
0 \longrightarrow & & & H^0(G,A) & \longrightarrow & H^0(G,B) & \longrightarrow & H^0(G,C) & \longrightarrow H^1(G,A) \longrightarrow \cdots
\end{array}
$$

The result then follows by applying the snake Lemma to the middle section of this diagram. $\square$

The next step is to show that we can use dimension shifting on this new long complex. This means we need to show that we can make a cohomologically and homologically trivial module as we did before in order to use dimension shifting. Now since we are interested only in Tate cohomology groups, we have that G is a finite group and as we remarked before, in this case we have an equivalent way of defining a coinduced module by using tensor products. From now on in this section we just refer to these modules as induced. Moreover, since we know that we can inject any G-module into an induced module, we can then use dimension shifting in both directions.

Proposition 1.47. *If M is an induced module, then for all $n \in \mathbb{Z}$, we have $\widehat{H}^n(G,M) = 0$*

Proof. Let $M = \mathrm{Coind}^G(A)$ for some abelian group G-module A. The easiest way to prove this is to use our original definition of the Tate cohomology groups as the the cohomology groups of $\widehat{C}^\bullet(G,A) = (X^\bullet)^G$. The idea is to show that

$$X^\bullet(G, \mathrm{Coind}^G(A)) \cong X^\bullet(G,A)$$

which we know is an exact sequence and therefore has trivial cohomology groups. Recall that $X^i(G,A) = \mathrm{Hom}(P_i, A)$. So in fact we want to show

$$\mathrm{Hom}_G(P_i, \mathrm{Coind}^G(A)) \cong \mathrm{Hom}(P_i, A),$$

but this is just Proposition 1.10 (c) with $H = 1$ and we can use it here since for all $i \in \mathbb{Z}$ we have that P_i is a G-module.

$\square$

Now with this, we can do dimension shifting on Tate cohomology groups as we did before for cohomology groups.

Lemma 1.48. *Let G be a finite group and let H be any subgroup, and let A be a G-module. Then given a G-module A, there exist G-modules A^+ and A^- such that for any subgroup H of G and any $n \in \mathbb{Z}$ we have*

$$\widehat{H}^{n-1}(H, A^-) \cong \widehat{H}^n(H, A) \cong \widehat{H}^{n+1}(H, A^+).$$

Proof. The same proof we used in 1.27 will work here since we can use induced modules in the same way as we did before.

$\square$

So with all this we can now use most of the results we had for cohomology groups and now use them for Tate cohomology groups.

1.7 Cup Products

In this section we establish a very important and useful construction that gives us a binary operation of the cohomology groups. This makes the cohomology groups into what is called a graded ring. Since we are mainly interested in Tate cohomology groups we will define the cup product for these groups, but we note here that this construction will work for all cohomology groups, not just Tate cohomology groups.

The motivation behind cup products is to define a map

$$\widehat{H}^n(G, A) \otimes \widehat{H}^m(G, B) \xrightarrow{\cup} \widehat{H}^{n+m}(G, A \otimes B).$$

Now in order to do this we first state the following elementary result from homological algebra.

Proposition 1.49. *Let A, A', B, B' be G-modules.*

(a) We can make $A \otimes B$ into a G-module by setting

$$g \cdot (a \otimes b) = g \cdot a \otimes g \cdot b$$

(b) There is a natural homomorphism

$$T : \operatorname{Hom}_{\mathbb{Z}[G]}(A, A') \otimes \operatorname{Hom}_{\mathbb{Z}[G]}(B, B') \longrightarrow \operatorname{Hom}_{\mathbb{Z}[G]}(A \otimes A', B \otimes B'),$$

which is defined by taking $\alpha \in \operatorname{Hom}_{\mathbb{Z}[G]}(A, A')$, $\beta \in \operatorname{Hom}_{\mathbb{Z}[G]}(B, B')$ and then defining $T(\alpha, \beta) = \alpha \otimes \beta$ to be the map

$$a \otimes b \mapsto \alpha(a) \otimes \beta(b), \qquad \textit{for } a \in A, b \in B,$$

and then extending bilinearly.

Proof. Both results are just simple verifications that the action and the map are both well-defined, but this is easy to see.

$\square$

If we apply Proposition 1.49 (b) to the P_i which make up the standard resolution of $\mathbb{Z}$, then we get a natural homomorphism

$$\operatorname{Hom}_G(P_i, A) \otimes \operatorname{Hom}_G(P_j, B) \longrightarrow \operatorname{Hom}_G(P_i \otimes P_j, A \otimes B).$$

Now the key to defining the cup product will be to find a G-homomorphism

$$\varphi_{i,j} : P_{i+j} \longrightarrow P_i \otimes P_j.$$

We will see why this is so important in a moment. Following [4], we have the following Theorem:

Theorem 1.50. *Let G be a finite group and let A, B be G-modules. Then there exists a unique family of homomorphisms*

$$\widehat{H}^n(G, A) \otimes \widehat{H}^m(G, B) \xrightarrow{\cup} \widehat{H}^{n+m}(G, A \otimes B), \qquad \textit{for any } n, m \in \mathbb{Z},$$

such that

(a) For $n = m = 0$, the cup product is induced by the natural map

$$A^G \otimes B^G \longrightarrow (A \otimes B)^G$$

(b) If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of G-modules such that

$$0 \longrightarrow A' \otimes B \longrightarrow A \otimes B \longrightarrow A'' \otimes B \longrightarrow 0$$

is also exact, then we have $(\delta a'') \cup b = \delta(a'' \cup b)$, where $a'' \in \widehat{H}^n(G, A''), b \in \widehat{H}^m(G, B)$ and δ denotes a general connection homomorphism between cohomology groups.

(c) If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is an exact sequence of G-modules such that

$$0 \longrightarrow A \otimes B' \longrightarrow A \otimes B \longrightarrow A \otimes B'' \longrightarrow 0$$

is exact, then we have $a \cup \delta b'' = (-1)^n \delta(a \cup b'')$, where $a \in \widehat{H}^n(G, A), b'' \in \widehat{H}^m(G, B'')$ and δ is again a general connection homomorphism between cohomology groups.

These homomorphisms are then called **cup products**

First we have to construct the $\varphi_{i,j}$ we mentioned before, for any $i, j \in \mathbb{Z}$. Once we have these G-homomorphisms, we can then define a map

$$\varphi'_{i,j} : \mathrm{Hom}_G(P_i \otimes P_j, A \otimes B) \longrightarrow \mathrm{Hom}_G(P_{i+j}, A \otimes B),$$
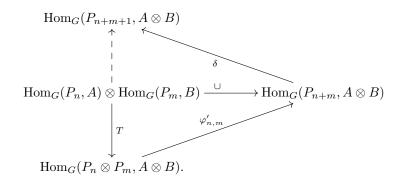$$(f \otimes g) \mapsto (f \otimes g) \circ \varphi_{i,j}.$$

With this we can then define the cup product as

$$f \cup g = \varphi'_{n,m} \circ T(f \otimes g) = (f \otimes g) \circ \varphi_{n,m},$$

with T as in Proposition 1.49.

Now since we want the cup product to be a map between the cohomology groups, we need $\varphi_{n,m}$ to induce a map between the cohomology groups and we need to see how it behaves with the boundary homomorphisms. To see this we first look at some of properties we want the cup product to satisfy.

Consider the following diagram:

$$\begin{array}{c}
\mathrm{Hom}_G(P_{n+m+1}, A \otimes B) \\
\\
\mathrm{Hom}_G(P_n, A) \otimes \mathrm{Hom}_G(P_m, B) \xrightarrow{\ \cup\ } \mathrm{Hom}_G(P_{n+m}, A \otimes B) \\
\\
\mathrm{Hom}_G(P_n \otimes P_m, A \otimes B).
\end{array}$$

Now the problem comes from the dotted arrow, since there are two ways it could be defined. We could first apply a boundary homomorphism to $\mathrm{Hom}_G(P_n, A)$ turning its elements into elements of $\mathrm{Hom}_G(P_{n+1}, A)$ and then

with this and $\text{Hom}_G(P_m, B)$ apply first T and then $\varphi_{n+1,m}$ to get to the top of the diagram. Alternatively we could of started with $\text{Hom}_G(P_m, B)$ and done the same procedure to also end up at the top. Since we would like the diagram to commute, this means that we want $\delta(f \cup g)$ to be the following

$$\delta(f \cup g) = (\delta f) \cup g + (-1)^n (f \cup (\delta g)).$$

This immediately tells us that cup products will send a pair of cocycles to a cocycle since, if f, g are cocycles, then $\delta f, \delta g$ will both be zero and we can clearly see from the definition of the cup product that this will mean $\delta(f \cup g)$ is also zero, meaning that $f \cup g$ is a cocycle.

Now from the definition of the cup product we see that that $\varphi_{n,m}$ must satisfy

$$\varphi_{n,m} \circ \partial = (\partial \otimes 1) \circ \varphi_{n+1,m} + (-1)^i (1 \otimes \partial) \circ \varphi_{n,m+1}; \tag{1}$$

$$(\epsilon \otimes \epsilon) \circ \varphi_{0,0} = \epsilon, \qquad (\epsilon = \text{augmentation map}). \tag{2}$$

Now to prove that cup products exist, we need to find $\varphi_{n,m}$ that satisfy $(1), (2)$ above. Once we have this, part (a) is just a consequence of (2) .

The second part of the proof is proving uniqueness, this is where parts (b) and (c) come in. First note that the sequences

$$0 \longrightarrow A \longrightarrow \text{Coind}^G(A) \longrightarrow A' \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow A'' \longrightarrow \text{Coind}^G(A) \longrightarrow A \longrightarrow 0$$

are split over $\mathbb{Z}$, so if we tensor them with any G-module B the result will still be exact and the middle module will still be induced, meaning it has trivial cohomology groups. So this combined with (b) and (c), allows us to use both cup products and dimension shifting. Uniqueness then follows by using (a) and then dimension shifting. The proofs of (b) and (c) together with the definition of the $\varphi_{n,m}$ can be found in [4] p. 106–107.

Now that we have defined cup products we state some of their properties.

Proposition 1.51. *(a)* $(f \cup g) \cup h = f \cup (g \cup h)$ *when considered as elements of* $\widehat{H}^{n+m+1}(G, A \otimes B \otimes C)$;

(b) $f \cup g = (-1)^{nm}(g \cup f)$ *modulo the identification* $A \otimes B = B \otimes A$;

(c) $\text{Res}(f \cup g) = \text{Res}(f) \cup \text{Res}(g)$;

(d) $\text{Cor}(f \cup \text{Res}(g)) = \text{Cor}(f) \cup g.$

Proof. See [9] Chapter V, Section 3.

$\square$

Finally we have one of the most important results related to cup products and Tate cohomology groups.

Lemma 1.52. *(Tate–Nakayama) Let G be a finite group, and let C be a G-module and $u \in H^2(G, C)$. If for every subgroup H of G we have that*

(i) $H^1(H, C) = 0$,

(ii) the size of $H^2(H, C)$ is the same as the size of H and is generated by $\text{Res}(u)$.

Then for any G-module M such that $\text{Tor}_1^{\mathbb{Z}}(M, C) = 0$, cup-product with u will give an isomorphism

$$\widehat{H}^i(G, M) \longrightarrow \widehat{H}^{i+2}(G, M \otimes_{\mathbb{Z}} C)$$

Proof. See [10] Chapter IX, Section 8. □

2   Class Field Theory

The intention of this section is to give a brief account of the main results in class field theory, so we do not include many proofs, but will only be showing the results that are needed to prove the main theorems of class field theory. To do this we will be following the approach of J.P. Serre [10] and J.S Milne [11].

2.1   Class Formation

We begin by defining the concept of a *formation*, which one can think of as an abstraction of the language of Galois theory.

Definition 2.1. A **formation** is a triple $(G, \{G_E\}_{E \in S}, A)$, where G is a group, S is some indexing set, and $\{G_E\}_{E \in S}$ is a family of subgroups of G of finite index such that:

1. If $G_E = G_F$, then $E = F$.

2. If E_i is a finite subset of S, then there exists some $F \in S$ such that $G_F = \bigcap G_{E_i}$.

3. If H is a subgroup of G that contains a subgroup G_E for some $E \in S$, then there exists some $F \in S$ such that $H = G_F$.

4. If $g \in G$ and $E \in S$, then $g \cdot G_E \cdot g^{-1} = G_F$ for some $F \in S$.

Finally A is a G-module such that for every $a \in A$, there exists some $E \in S$ with $G_E = \text{Stab}(a)$ (the stabilizer of a), and if $F \in S$, the we set A_F to be the subgroup of A consisting of the elements fixed by G_F.

Now to a class formation we apply all the terminology of Galois theory. So we call the elements of S, "fields" and we say $E \subset F$ if $G_F \subset G_E$. Also we say F/E is a Galois extension if G_F is a normal subgroup of G_E and we let $[F : E] = |G_E/G_F|$.

As a consequence of this definition we have $H^0(G_E, A) = A_E$, and if F/E is Galois, then $G(F/E) := G_E/G_F$ will act on A_F, and $H^0(G(F/E), A_F) = A_E$. The thing to note, is that even though we are using the language of Galois theory, this notion of a formation will work for any topological group G, with $\{G_E\}$ consisting of all the closed subgroups of finite index. So this allows us to use our intuition from Galois theory, and apply it to a different class of objects.[2]

Definition 2.2. A **class formation** consists of a formation $(G, \{G_E\}_{E \in S}, A)$ together with an injective homomorphism

$$\text{inv}_E : H^2(G(F/E), A_F) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

for each Galois extension F/E, and satisfying the following conditions:

1. For every Galois extension F/E, we have $H^1(G(F/E), A_F) = 0$;

2. The homomorphism inv_E sends $H^2(G(F/E), A_F)$ to the unique subgroup of $\mathbb{Q}/\mathbb{Z}$ of order $[F : E]$.

3. For any extension E'/E, we have

$$\text{inv}_{E'} \circ \text{Res}_{E/E'} = [E' : E]. \text{inv}_E,$$

where Res is the usual restriction map.

---

[2]One fo the differences between a formation and Galois theory is that we don't need to have a one-to-one correspondence $G_E \leftrightarrow A_E$

If F/E is a Galois extension of degree n, then there is a unique element $u_{F/E} \in H^2(G(F/E), A_F)$ such that $\text{inv}_E(u_{F/E}) = 1/n$. From the above it is clear that $u_{F/E}$ must generate $H^2(G(F/E), A_F)$, and we call this element the **canonical or fundamental class** of the extension F/E.

Now in order to use this for class field theory, we need to show that in both the local and global case, we can find an appropriate A that can be used to make a class formation. Once we have this then we will apply the Tate–Nakayama Lemma, to get what is known as the Artin reciprocity map.

2.2 Local Class Field Theory

In order to ease notation, throughout this section we write $H^n(G(F/E), F^\times) = H^n(F/E)$.

Definition 2.3. The **Brauer group** B_F of a field F is defined to be $H^2(F_s/F)$, where F_s denotes a chosen separable closure of F. In fact, B_F is the direct limit of $H^2(E/F)$ as E runs through all the finite Galois extensions of F.

In this this section we let K denote a field, that is complete with respect to some discrete valuation v, and we denote by k its residue field which we assume to be perfect. In this setting one can prove that B_K is the direct limit of the subgroups $H^2(E/K)$, as E runs through all the finite unramified Galois extensions of K (see [10], Proposition 3, p. 184).

Definition 2.4. Let K be a field, with K_s a chosen separable algebraic closure of K. Then we say an element $f \in \text{Gal}(K_s/K)$ provides K with a structure of a **quasi-finite field** if the following hold:

1. K is a perfect field

2. the map $\widehat{\mathbb{Z}} \longrightarrow \text{Gal}(K_s/K)$ sending $v \mapsto f^v$ is an isomorphism.

Theorem 2.5. *Let K be a field that is complete with respect to a discrete valuation v, and has perfect residue field k. Then there is a exact sequence*

$$0 \longrightarrow B_k \longrightarrow B_K \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

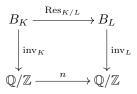*Proof.* Use [10] Theorem 2, p. 186 and Proposition 1, p. 189.

$\square$

Now it turns out that the when k (the residue field of K) is a quasi-finite field, then its Brauer group will be zero (See [10] Proposition 5, p. 192), and with this and Theorem 2.5 we have:

Proposition 2.6. *If K is a field that is complete with respect to discrete valuation v, and with quasi-finite residue field k, then we have an isomorphism $B_K \longrightarrow \mathbb{Q}/\mathbb{Z}$, which we denote by inv_K.*

Proof. Use the exact sequence from Theorem 2.5 and the fact that $B_k = 0$ to get the isomorphism we seek.

$\square$

With this isomorphism in hand we can now state one of the key results that will later allow us to make a class formation in the local setting.

Proposition 2.7. *Let L be a finite extension of K, with $n = [L : K]$, and let $\text{Res}_{K/L} : B_K \longrightarrow B_L$ denote the standard restriction map between cohomology groups. Then*

$$
\begin{array}{ccc}
B_K & \xrightarrow{\text{Res}_{K/L}} & B_L \\
\downarrow{\scriptstyle \text{inv}_K} & & \downarrow{\scriptstyle \text{inv}_L} \\
\mathbb{Q}/\mathbb{Z} & \xrightarrow{\quad n \quad} & \mathbb{Q}/\mathbb{Z}
\end{array}
$$

is a commutative diagram.

Proof. See [10] Proposition 7, p. 193.

$\square$

We can now use these results to show how to make a class formation. We start by making a formation and then using the above to show it satisfies the conditions to be a class formation.

We begin by choosing a separable closure K_s of K (here K has residue field k, which is quasi-finite), and letting S be the set of all finite subextensions of K_s. We let $G = \mathrm{Gal}(K_s/K)$ and for $E \in S$, we set $G_E = \mathrm{Gal}(K_s/E)$. Now by using Galois theory, it is easy to see that we have a formation $(G, \{G_E\}_{E \in S}, K_s^\times)$. Also for any $E \in S$, the residue field of E will be a finite extension of k and therefore will be canonically provided with the structure of a quasi-finite field.

Next we need to show that this formation is actually a class formation. First recall that $B_E = H^2(K_s/E)$, so Proposition 2.6 gives us an isomorphism

$$\mathrm{inv}_E : H^2(\mathrm{Gal}(K_s/E), K_s^\times) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

for each $E \in S$. As for the three conditions we have on a class formation, we have to check

1. For every Galois extension F/E, we have $H^1(F/E) = 0$;

2. The homomorphism inv_E sends $H^2(F/E)$ to the unique subgroup of $\mathbb{Q}/\mathbb{Z}$ of order $[F:E]$;

3. For any extension E'/E, we have
$$\mathrm{inv}_{E'} \circ \mathrm{Res}_{E/E'} = [E':E]\,\mathrm{inv}_E\,.$$

But this is where the results above come in. The first condition is just the standard "Hilbert Theorem 90", and the other two conditions are corollaries of Proposition 2.7.

Now that we know we have a class formation, we know, for example, that for any finite Galois extension F/E, there is a unique element $u_{F/E} \in H^2(F/E)$, such that $\mathrm{inv}_E(u_{F/E}) = 1/[F:E]$. We call $u_{F/E}$ the fundamental (or canonical) class of the extension F/E (later on, in order to differentiate between cocycles and the class they represent, we will denote this as $\overline{u_{F/E}}$). One of the most important consequences of having a class formation is that we can use the Tate–Nakayama Lemma to get the following result:

Proposition 2.8. *For any $n \in \mathbb{Z}$, taking cup product with the fundamental class $u_{F/E}$, defines an isomorphism*

$$\widehat{H}^n(\mathrm{Gal}(F/E), \mathbb{Z}) \longrightarrow \widehat{H}^{n+2}(\mathrm{Gal}(F/E), F^\times).$$

Proof. Take $C = F^\times$ and $u = u_{F/E}$ in Lemma 1.52.

$\square$

In particular, the case $n = -2$ gives us an isomorphism

$$\alpha_{F/E} : \widehat{H}^{-2}(\mathrm{Gal}(F/E), \mathbb{Z}) \longrightarrow \widehat{H}^0(\mathrm{Gal}(F/E), F^\times).$$

Now the first group is nothing more than $\mathrm{Gal}(F/E)^{ab}$ and the second group, by definition, is simply $E^\times/N_G F^\times$. The inverse of this isomorphism is known as the Artin reciprocity map, and its existence is one of the most important results in local class field theory.

2.3 Global Class Field Theory

In the global case the situation is not as simple. In order to make a class formation, we have to define a new module on which the Galois groups will act.

Definition 2.9. Let K be a global field, and let K_v denote the completion of K with respect to a valuation v. The **adele ring** $\mathbb{A}_K$ of K, is a subring of $\prod_v K_v$, where v runs through all the places [3] of K and for $\alpha = (\alpha_v) \in \mathbb{A}_K$, we have $\alpha_v \in \mathcal{O}_v$ for all but finitely many v (here $\mathcal{O}_v$ denotes the ring of integers of K_v, for v non-archimedean). We make $\mathbb{A}_K$ into a ring by defining the addition and multiplication componentwise. The construction of $\mathbb{A}_K$ from K_v as above, is called the *restricted* product of K_v with respect to the subrings $\mathcal{O}_v$.

If instead the take the restricted topological product of $K_v^\times$ with respect to $\mathcal{O}_v^\times$, then we get the **idele group** J_K of K, which is just the unit group of the adeles, so we have $J_K = \mathbb{A}_K^\times$. With this definition, $\mathbb{A}_K$ will be a locally compact topological group, since the K_v are locally compact and the $\mathcal{O}_v$ are compact. Also since for each place v of K we have a natural inclusion $K \subset K_v$, so we can embed K diagonally into $\mathbb{A}_K$, and hence we get a diagonal embedding

$$K^\times \longrightarrow J_K.$$

Thus we can view $K^\times$ as a subgroup of J_K, and finally with this we can define the **idele class group** $C_K = J_K / K^\times$.

As before we want to first make a formation and then show that we can turn it into a class formation. From Galois theory we already have most components of the formation, we just just need to find a module on which the Galois groups act, such that we get a class formation. This is where the idele class group C_K comes in, it will play the same role $K^\times$ played in the local setting.

With this in mind take L/K a finite Galois extension with Galois group $G_{L/K} = \text{Gal}(L/K)$, then $G_{L/K}$ will naturally act on J_L. To see how it acts, first note there is a natural inclusion of J_K into J_L. So take $\mathfrak{p}$ to be a prime in K and $\mathfrak{P}$ to be a prime in L lying above $\mathfrak{p}$ (i.e. $\mathfrak{P} \mid \mathfrak{p}$), then for any $g \in G_{L/K}$ it is not hard to show that $g\mathfrak{P} \mid \mathfrak{p}$, and furthermore for any $\mathfrak{P}, \mathfrak{P}'$, lying over $\mathfrak{p}$, there exists $g \in \text{Gal}_{L/K}$ such that $g\mathfrak{P} = \mathfrak{P}'$. From this we have an induced action of $G_{L/K}$ on C_L.

Proposition 2.10. *If L/K is a Galois extension with Galois group $G_{L/K}$, then $C_L^{G_{L/K}} = C_K$.*

Proof. See [12] Proposition (2.5), p. 372.

$\square$

Now we can use this to make a formation. We begin with a global field K and a chosen separable algebraic closure K_s of K. Let $G = \text{Gal}(K_s/K)$. Next let S be the set of all finite extensions of K in K_s. For any $E \in S$ we define $G_E = \text{Gal}(K_s/E)$, and finally we let

$$C = \varinjlim_{E \in S} C_E$$

(the direct limit). Then we have that $(G, \{G_E\}_{E \in S}, C)$ is a formation.

To show that we can make this into a class formation, it suffices to show that for any finite Galois extension L/K with Galois group G, the following hold:

1. $H^1(G, C_L) = 0$;

2. The group $H^2(G, C_L)$ is cyclic of order $[L : K]$, with a canonical generator $u_{L/K}$;

3. If L, E are finite Galois extensions of K with $L \subset E$, then

$$\text{Res} : H^2(E/K) \longrightarrow H^2(E/L)$$

---

[3]By place we mean an equivalence class of valuations

sends $u_{E/K}$ to $u_{L/K}$.

(See [13], p. 151–155.) In order to prove that these conditions hold, we use the results of local class field theory, but even with this, there is still a lot of work to be done. The idea is to take a Galois extension L/K with Galois group $G_{L/K}$. Then one can show that for any $n \in \mathbb{Z}$,

$$\widehat{H}^n(G_{L/K}, J_L) \cong \bigoplus_v \widehat{H}^n(G_{L/K}^v, (L^v)^\times).$$

Here the sum is taken over all primes v of K, and for some $w \mid v$, we have $G_{L/K}^v$ being the decomposition group of w, and $L^v = L_w$ (the choice of $w \mid v$ does not matter, as all the decomposition groups will be isomorphic and similarly all the fields L_w will be isomorphic as long as $w \mid v$). This allows us to use the results from the local case. To complete the proof, one must show the following:

(a) The Herbrand quotient

$$h(C_L) = \frac{\mid \widehat{H}^0(G_{L/K}, C_L) \mid}{\mid H^1(G_{L/K}, C_L) \mid},$$

is equal to $[L : K]$.

(b) The group $H^1(G_{L/K}, C_L)$ is trivial.

(c) The group $H^2(G_{L/K}, C_L)$ is finite and its order divides $[L : K]$;

(d) The inv maps defined in the local case, define a map from $H^2(G_{L/K}, J_L)$ to $\mathbb{Q}/\mathbb{Z}$;

(e) The group $H^2(G_{L/K}, C_L)$ is a cyclic group of order $[L : K]$;

(f) For L, K finite extension of K, with $L \subset E$, $\mathrm{Res}(u_{E/K}) = u_{L/K}$.

For a proof of these facts see [14] p. 180–203 or [13] p. 151–155.

Once we have this, then as before, we can use the Tate–Nakayama Lemma to get an isomorphism

$$\widehat{H}^{-2}(G_{L/K}, \mathbb{Z}) \xrightarrow{\sim} \widehat{H}^0(G_{L/K}, C_K),$$

but note that the first group is $G_{L/K}^{ab}$ and the second group is just $C_K / N_{G_{L/K}} C_L$. The inverse of this isomorphism is called the Artin reciprocity map.

3   Topological Groups

In this section, we look at the notion of a topological group. As in the previous section, we will not give many proofs, as the intention is only to show some of the properties of topological groups, since we will be working with them in the next section.

Definition 3.1. A **topological group** G is a topological space, with a group structure such that the multiplication map and inversion map in the group are continuous with respect to the topology on G.

Note that we can make any group into a topological group by simply giving it the discrete topology, but we cannot make any topological space into a topological group.

The fact that in a topological group, the multiplication map is continuous, means that we can look at a property of a topological group at a point, and then use the multiplication and inversion map to show that at every point such a property holds.

Proposition 3.2. *If G is a topological group, then the following hold:*

(a) If H is an open (resp. closed) subgroup of G, then every coset of H will be open (resp. closed);

(b) Every open subgroup of G is closed in G, and every closed subgroup of finite index in G is open;

(c) The topological group G is Hausdorff if and only if $\{1\}$ is a closed subgroup of G;

(d) If H is a normal subgroup of G, then H is closed in G if and only if G/H is Hausdorff.

Proof. See [15] Lemma 0.3.1, p. 6.

$\square$

Next we prove some results that we will be using in the next section.

Proposition 3.3. *Let G be a topological group, and let H be a normal subgroup of G. Then H is open in G if and only if the quotient topology on G/H is discrete.*

Proof. Take $x \in G/H$, then under the natural map $\varphi : G \longrightarrow G/H$, we have that $\varphi^{-1}(x) = xH$ (the coset of H by x), now this coset will be open if and only if H is open (by (a) above), and from this the result follows at once.

$\square$

Lemma 3.4. *Let A be a topological group and let H and S be subgroups of A with H open in A. If $H \cap S$ is closed in H then S is closed in A.*

Proof. Since H is a subgroup of A we can write it as the disjoint union of the cosets $\{Ha_i \mid i \in I\}$ for some indexing set I. Here each coset is open since H is open and therefore they are all closed as well (by (b) above). Hence as a topological space, A is the disjoint union of these topological spaces Ha_i. So in order to prove that S is closed in A it suffices to prove that $S \cap Ha_i$ is closed in Ha_i for all $i \in I$.

First assume that $S \cap Ha_i \neq \emptyset$, then we can pick $\sigma \in S \cap Ha_i$, and we can write $Ha_i = H\sigma$. Now since we are in a topological group the map given by multiplication by σ is a homeomorphism, we can apply this to $S \cap H$ to get that $S\sigma \cap H\sigma = S \cap H\sigma$ is closed in $H\sigma$ and hence the result. If $S \cap Ha_i = \emptyset$, then $S \cap Ha_i$ is closed in Ha_i, since the empty set is always closed.

$\square$

4 Langlands' Correspondence for Algebraic Tori

In what follows we have tried to stay faithful to the main ideas in Langlands' paper [1], but aim to present the results in more detail and in more generality, by considering representations into any divisible abelian topological group.

4.1 Setup and Notation

We begin by noting that there is one-to-one correspondence between algebraic tori defined over a field F, that split over a finite Galois extension K of F, and equivalence classes of lattices on which $\mathrm{Gal}(K/F)$ acts. Here by lattice we mean a finitely generated free $\mathbb{Z}$-module (i.e. isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{Z}_{\geq 0}$). If T is an algebraic torus and it corresponds to the lattice L, then the group $T(K)$ of K-rational points corresponds to the $\mathrm{Gal}(K/F)$-module $\mathrm{Hom}(L, K^\times)$. Moreover, we have that $T(K)^{\mathrm{Gal}(K/F)} = T(F)$ (see [16], Chapter III).

Next we make the following definitions which we will use throughout. Let $\widehat{L} = \mathrm{Hom}(L, \mathbb{Z})$ and $\widehat{T}_D = \mathrm{Hom}(\widehat{L}, D)$, where D is some fixed divisible abelian topological group with trivial $\mathrm{Gal}(K/F)$-action.

From the action of $\mathfrak{G} = \mathrm{Gal}(K/F)$ on L, we can define a $\mathfrak{G}$-action on $\widehat{L}$ by setting

$$(g\lambda)(x) = \lambda(g^{-1} \cdot x),$$

for $\lambda \in \widehat{L}$, $x \in L$ and $g \in \mathfrak{G}$. Similarly we define the action of $\mathfrak{G}$ on $\widehat{T}_D$ as

$$(g\alpha)(\lambda) = \alpha(g^{-1} \cdot \lambda)$$

where $\alpha \in \widehat{T}_D$, $\lambda \in \widehat{L}$ and $g \in \mathfrak{G}$.

Notation. From now on we make the following notational conventions:

1. We let F be any field and K a finite Galois extension of K, and we denote $\mathrm{Gal}(K/F)$ simply by $\mathfrak{G}$.

2. If G is any group and A is a G-module, then for $x \in Z_n(G, A)$, we let $\overline{x}$ or $[x]$ represent its class in $H_n(G, A)$, and we do the same for cohomology and Tate cohomology groups.

3. If A, B are two topological groups, we let $\mathrm{Hom}_{cts}(A, B)$ represents the group of continuous group homomorphisms from A to B, and similarly $Z^1_{cts}(A, B), B^1_{cts}(A, B)$ represent the continuous 1-cocycles and 1-coboundaries, respectively, and $H^1_{cts} = Z^1_{cts}/B^1_{cts}$.

4.2 The Weil Group

Following [17], we define the Weil group of a local or global field and state some of its properties.

Definition 4.1. The **Weil group** of a local or global field F, is a triple $(W_F, \varphi, \{r_E\})$, where W_F is a topological group, φ is a continuous homomorphism from W_F to $\mathrm{Gal}(F_s/F)$ (where F_s denotes a chosen separable closure of F) such that the image of φ in $\mathrm{Gal}(F_s/F)$ is dense. For each finite extension $E \subset F_s$ of F, we let $W_E = \varphi^{-1}(\mathrm{Gal}(F_s/E))$, and

$$C_E = \begin{cases} \text{the idele class group of } E & \text{if } E \text{ is a global field} \\ E^\times & \text{if } E \text{ is a local field.} \end{cases}$$

Furthermore, associated to each field E, we have an isomorphism

$$r_E : C_E \longrightarrow W_E^{ab},$$

where W_E^{ab} is the quotient of W_E by the closure of its commutator subgroup which is denoted by W_E^c. Moreover, for any finite extension K/F (not necessarily Galois) we can define its Weil group as
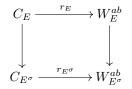
$$W_{K/F} = W_F/W_K^c.$$

Lastly, we need the following conditions to hold:

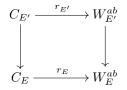1. For each E, the map

$$C_E \longrightarrow G_E^{ab},$$

induced from r_E and $W_E^{ab} \longrightarrow G_E^{ab}$, is the Artin reciprocity map as in section 2.

2. Let $w \in W_F$ and $\sigma = \varphi(w) \in G_F$. Then for each E, we have the following commutative diagram

$$\begin{array}{ccc} C_E & \xrightarrow{\ r_E\ } & W_E^{ab} \\ \downarrow & & \downarrow \\ C_{E^\sigma} & \xrightarrow{\ r_{E^\sigma}\ } & W_{E^\sigma}^{ab} \end{array}$$

where the first vertical arrow, it the map induced by σ and the second vertical arrow is the map given by conjugation by w.

3. For $E' \subset E$, we have the following commutative diagram:

$$\begin{array}{ccc} C_{E'} & \xrightarrow{\ r_{E'}\ } & W_{E'}^{ab} \\ \downarrow & & \downarrow \\ C_E & \xrightarrow{\ r_E\ } & W_E^{ab} \end{array}$$

where the first vertical arrow is just the natural map induced by the inclusion, and the second vertical arrow is the Transfer map as defined in 1.36

4. The natural map

$$W_F \longrightarrow \varprojlim_E W_{E/F}$$

is an isomorphism of topological groups.

For our purposes we have a simple description of the Weil group as follows:

Proposition 4.2. *If K/F is a Galois extension of F, then we have an exact sequence*

$$0 \longrightarrow C_K \longrightarrow W_{K/F} \xrightarrow{\ \sigma\ } \mathrm{Gal}(K/F) \longrightarrow 0,$$

which gives us an element in $H^2(G, C_K)$, whose class is the fundamental class $\overline{u_{K/F}}$ as in section 2.

Proof. See [17], (1.2), p. 4. □

We fix once and for all $\{w_g \mid g \in \mathfrak{G}\}$ to be the set of left coset representatives of C_K in $W_{K/F}$.

Proposition 4.3. *If F is a global field, then for each place v of F we have a commutative diagram*

$$\begin{array}{ccc} W_{F_v} & \xrightarrow{\ \varphi_v\ } & \mathrm{Gal}((F_v)_s/F_v) \\ \downarrow & & \downarrow \\ W_F & \xrightarrow{\ \varphi\ } & \mathrm{Gal}(F_s/F) \end{array}$$

for details see [17], Proposition (1.6.1), p. 8.

In this paper we will be concerned with representations of $W_{K/F}$ into the group

$$^L T_D = \widehat{T}_D \rtimes \mathrm{Gal}(K/F)$$

(when $D = \mathbb{C}^\times$, this group is known as the L-group of T). We want to study continuous homomorphisms

$$\phi : W_{K/F} \longrightarrow \widehat{T}_D \rtimes \mathrm{Gal}(K/F)$$

that make

$$
\begin{array}{ccc}
W_{K/F} & \xrightarrow{\ \sigma\ } & \mathrm{Gal}(K/F) \\
\Big\downarrow{\scriptstyle \phi} & & \Big\| \\
\widehat{T}_D \rtimes \mathrm{Gal}(K/F) & \longrightarrow & \mathrm{Gal}(K/F)
\end{array}
$$

a commutative diagram; these are called admissible homomorphisms. Two admissible homomorphisms α, β from $W_{K/F}$ to $^L T_D$ are called equivalent if there exists $t \in \widehat{T}_D$ such that $\alpha = t\beta t^{-1}$. Now note that we can write $\phi = f \times \sigma$, where $f \in Z^1_{cts}(W_{K/F}, \widehat{T}_D)$, from which it follows that two admissible homomorphisms

$$\alpha = f_\alpha \times \sigma \qquad \text{and} \qquad \beta = f_\beta \times \sigma$$

from $W_{K/F}$ to $^L T_D$ are equivalent if and only if f_α and f_β represent the same cohomology class of $H^1_{cts}(W_{K/F}, \widehat{T}_D)$.

4.3   The Duality Theorem

With the setup above, we are in a position to state the first of the two main theorems we are going to prove.

Theorem 4.4. *If K is a local or global field, then there is a canonical isomorphism of $H^1_{cts}(W_{K/F}, \widehat{T}_D)$ with $\mathrm{Hom}_{cts}(\mathrm{Hom}_{\mathrm{Gal}(K/F)}(L, C_K), D)$.*

We begin by proving that there exists an isomorphism

$$\Psi : H^1(W_{K/F}, \widehat{T}_D) \longrightarrow \mathrm{Hom}(\mathrm{Hom}_{\mathrm{Gal}(K/F)}(L, C_K), D),$$

and then we prove that $\Psi(\overline{f})$ is a continuous homomorphism if and only if $f \in Z^1_{cts}(W_{K/F}, \widehat{T}_D)$.

We begin by noting that we can extend the natural action of $\mathfrak{G}$ on C_K, to that of $W_{K/F}$ on C_K, by letting $W_{K/F}$ act by conjugation, i.e., if $w \in W_{K/F}$ and $a \in C_K$, then

$$w \cdot a = waw^{-1}.$$

So that in this case, since C_K is an abelian normal subgroup of $W_{K/F}$, we see that C_K will act trivially on itself and hence we get an induced $\mathfrak{G}$ action, which agrees with the standard Galois action of $\mathfrak{G}$ on C_K. Also all $\mathfrak{G}$-modules can be viewed as $W_{K/F}$ modules, and therefore can also be viewed as C_K-modules, where C_K will act trivially.

Remark: In what follows we will be proving results about homology groups, and in the proofs we will always work with n-cycles and usually ignore n-boundaries, since in all of these cases the maps involved are maps between homology groups which will automatically send boundaries to boundaries, so all that we need to check is how the maps in question act on the n-cycles.

Proposition 4.5. *There is a natural $\mathfrak{G}$-isomorphism of $H_1(C_K, \widehat{L})$ with $\mathrm{Hom}(L, C_K)$.*

Proof. Since C_K acts trivially on $\widehat{L}$ we have that

$$H_1(C_K, \widehat{L}) \cong C_K \otimes_{\mathbb{Z}} \widehat{L}$$

since it is a standard result in homological algebra that if a group X acts trivially on a X-module A, then $H_1(X, A) \cong X/[X, X] \otimes_{\mathbb{Z}} A$ where $[X, X]$ denotes the commutator subgroup (see [5] p. 164). So in this case, since C_K is an abelian group we get the result above and note this will be a $\mathfrak{G}$-isomorphism[4]. Furthermore we have a natural $\mathfrak{G}$-isomorphism

$$C_K \otimes_{\mathbb{Z}} \widehat{L} \longrightarrow \mathrm{Hom}(L, C_K)$$

where for $\widehat{\lambda} \in \widehat{L}$ and $a \in C_K$ we send $a \otimes \widehat{\lambda}$ to the homomorphism $\lambda \longrightarrow a^{\langle \lambda, \widehat{\lambda} \rangle}$ where $\lambda \in L$ and $\langle -, - \rangle$ is the natural bilinear paring $\langle -, - \rangle : L \times \widehat{L} \longrightarrow \mathbb{Z}$, and this is a $\mathfrak{G}$-isomorphism. Combining these two isomorphisms, we get a $\mathfrak{G}$-isomorphism

$$H_1(C_K, \widehat{L}) \longrightarrow \mathrm{Hom}(L, C_K)$$

<div align="right">□</div>

Under this isomorphism we see that a 1-cycle $x \in Z_1(C_K, \widehat{L})$, will map to the homomorphism

$$\lambda \longmapsto \prod_{a \in C_K} a^{\langle \lambda, x(a) \rangle}, \qquad \text{for } \lambda \in L.$$

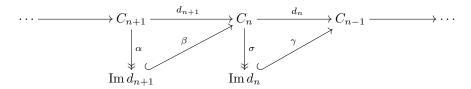Note the homomorphism makes sense as the support of the 1-cycles and 1-boundaries is always finite.

Next we need a couple of results in homological algebra in order to simplify the task of finding the isomorphism of Theorem 4.4.

Lemma 4.6. *Let R, A be any rings, $C^{\bullet}$ a complex of R-modules, and let T be an exact contravariant functor from R-modules to A-modules. Then*
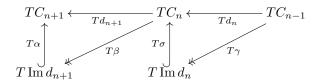
$$TH_n(C^{\bullet}) = H_n(TC^{\bullet}), \qquad \text{for all } n \in \mathbb{Z}_{\geq 0}.$$

In words, this Lemma simply says that exact contravariant functors commute with homology.

Proof. From the complex $C^{\bullet}$ we can form the following diagram.

First we are going to use this to show that $\ker(Td_{n+1}) = T(\mathrm{coker}(d_{n+1}))$ and $\mathrm{Im}(Td_n) = T(\mathrm{Im}(d_n))$, from which we will then deduce the result. We begin by applying T to this diagram and noting that a contravariant functor sends injective maps to surjective maps and vice-versa, which gives the following commutative diagram.

Here the first triangle tells us that $\ker(Td_{n+1}) = \ker(T\beta)$ and the second tells us that $\mathrm{Im}(Td_n) = \mathrm{Im}(T\sigma)$. Now for $T\beta$ and $T\sigma$ we have the following exact sequences

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[4]Here $\mathfrak{G}$ acts on 1-cycles $x \in Z_1(C_K, \widehat{L})$ as $g \cdot x(a) = gx(g^{-1} \cdot a)$ for all $a \in C_K$, and $\mathfrak{G}$ acts diagonally on $C_K \otimes \widehat{L}$

$$0 \longrightarrow T(\operatorname{coker}(d_{n+1})) \longrightarrow TC_n \xrightarrow{T\beta} T(\operatorname{Im} d_{n+1}) \longrightarrow 0,$$

$$0 \longrightarrow T(\operatorname{Im}(d_n)) \xrightarrow{T\sigma} TC_n \longrightarrow T(\ker d_n) \longrightarrow 0,$$

and the top sequence tells us that $\ker(T\beta) = T(\operatorname{coker}(d_{n+1}))$ and the bottom tells us that $\operatorname{Im}(T\sigma) = T(\operatorname{Im}(d_n))$. So the first part is done and now we use this to prove the result. Consider the following exact sequence

$$0 \longrightarrow H_n(C^\bullet) \longrightarrow C_n/\operatorname{Im}(d_{n+1}) \longrightarrow C_n/\ker(d_n) \longrightarrow 0,$$

if we then apply T to this exact sequence we get

$$0 \longrightarrow T(C_n/\ker(d_n)) \longrightarrow T(C_n/\operatorname{Im}(d_{n+1})) \longrightarrow T(H_n(C^\bullet)) \longrightarrow 0,$$

but this tells us that

$$T(H_n(C^\bullet)) = \frac{T(C_n/\operatorname{Im}(d_{n+1}))}{T(C_n/\ker(d_n))}.$$

Note that the group on the top is just $T(\operatorname{coker}(d_{n+1}))$ and the bottom group is $T(\operatorname{Im}(d_n))$, so using the first part of the proof, we can write this as

$$T(H_n(C^\bullet)) = \ker(Td_{n+1})/\operatorname{Im}(Td_n) = H_n(TC^\bullet)$$

and hence we have the result.

$\square$

Proposition 4.7. *Let G be any group, and let D be a divisible abelian group with trivial G-action. Then for all $n > 0$ we have an isomorphism $H^n(G, \operatorname{Hom}(B, D)) \longrightarrow \operatorname{Hom}(H_n(G, B), D)$, for any left G-module B.*

Proof. It is a standard result in homological algebra that for any two rings R, S and any right R, S modules A, C, respectively, with B a $R - S$-bimodule, we have a natural isomorphism

$$\operatorname{Hom}_S(A \otimes_R B, C) \xrightarrow{\sim} \operatorname{Hom}_R(A, \operatorname{Hom}_S(B, C))$$

(see [6] Theorem 2.75). So if we let $S = \mathbb{Z}$, $R = \mathbb{Z}[G]$ and $C = D$, then we get

$$\operatorname{Hom}_\mathbb{Z}(A \otimes_{\mathbb{Z}[G]} B, D) \cong \operatorname{Hom}_G(A, \operatorname{Hom}_\mathbb{Z}(B, D)).$$

Now we are going to replace A by a projective resolution $X^\bullet$ of $\mathbb{Z}$ by G-modules. In this case we can consider the X^i as both right or left modules by using the usual trick of introducing inverses, and the reason such a projective resolution exists is due to the fact that category of G-modules has enough projectives. So the above gives rise to isomorphic complexes

$$\operatorname{Hom}_\mathbb{Z}(X^\bullet \otimes_{\mathbb{Z}[G]} B, D) \cong \operatorname{Hom}_G(X^\bullet, \operatorname{Hom}_\mathbb{Z}(B, D)).$$

Now we take homology of both sides. On the left we can use Lemma 4.6 which tells us that for a contravariant functor T, we have $H_n(TC^\bullet) = TH_n(C^\bullet)$ for any complex $C^\bullet$, so since $\operatorname{Hom}_\mathbb{Z}(-, D)$ is an exact contravariant functor (as D is $\mathbb{Z}$-injective), the homology in the left is $\operatorname{Hom}_\mathbb{Z}(H_n(G, B), D)$. For the complex on the right taking homology, we get $H^n(G, \operatorname{Hom}_\mathbb{Z}(B, D))$ and since the complexes are isomorphic, the homology groups will be isomorphic. $\square$

Remark 4.8. *For $n = 1$, the isomorphism from Proposition 4.7 can be seen to be induced by the paring*

$$H^1(G, \mathrm{Hom}(B, D)) \times H_1(G, B) \longrightarrow D,$$

which sends a 1-cocycle f and a 1-cycle x to[5]

$$\sum_{g \in G} \langle f(g), x(g) \rangle.$$

We can now use this result to reduce the task of finding an isomorphism

$$\Psi : H^1(W_{K/F}, \widehat{T}_D) \longrightarrow \mathrm{Hom}(\mathrm{Hom}_{\mathrm{Gal}(K/F)}(L, C_K), D),$$

to finding an isomorphism

$$H_1(W_{K/F}, \widehat{L}) \overset{\sim}{\longrightarrow} H_1(C_K, \widehat{L})^{\mathfrak{G}},$$

since once we have this, setting $n = 1$, $B = \widehat{L}$, and $G = W_{K/F}$ in Proposition 4.7 and using Proposition 4.5 we get the isomorphism we seek.

Now to find this isomorphism we use the fact C_K is a normal subgroup of $W_{K/F}$ and of finite index $|\mathfrak{G}|$, which means we can use the transfer map to give us homomorphisms

$$\mathrm{Tr}_n : H_n(W_{K/F}, \widehat{L}) \longrightarrow H_n(C_K, \widehat{L})$$

(see 1.36). One of our goals is to show that, in fact,

$$\mathrm{Tr}_1 : H_1(W_{K/F}, \widehat{L}) \longrightarrow H_1(C_K, \widehat{L})^{\mathfrak{G}}$$

is an isomorphism.

Since we are working with the idele class group C_K, the Tate–Nakayama Lemma tells us that we can use cup products to obtain isomorphisms between Tate cohomology groups as long as the conditions for the Tate–Nakayama Lemma are satisfied. From class field theory (see section 2) we know that 1.52 $(i), (ii)$ hold for $C = C_K$ and $u = \overline{u_{K/F}}$. So we need only check that $\mathrm{Tor}_1^{\mathbb{Z}}(\widehat{L}, C_K) = 0$. This is true since $\widehat{L}$ is a free and hence flat $\mathbb{Z}$-module, and it is a standard result that this is equivalent to $\mathrm{Tor}_1^{\mathbb{Z}}(\widehat{L}, M) = 0$ for any $\mathbb{Z}$-module M (see [3] p. 790). So we can use Tate–Nakayama to form the following diagram.

$$\widehat{H}^{-2}(\mathfrak{G}, \widehat{L})$$
$$\|$$

$$H_1(C_K, \widehat{L}) \xrightarrow{\mathrm{Cor}} H_1(W_{K/F}, \widehat{L}) \xrightarrow{\mathrm{Coinf}} H_1(\mathfrak{G}, \widehat{L}) \longrightarrow 0$$
$$\downarrow{N_{\mathfrak{G}}} \qquad\qquad \downarrow{\mathrm{Tr}_1} \qquad\qquad \downarrow{\cong}$$
$$0 \longrightarrow N_{\mathfrak{G}}(H_1(C_K, \widehat{L})) \lhook\joinrel\longrightarrow H_1(C_K, \widehat{L})^{\mathfrak{G}} \longrightarrow \widehat{H}^0(\mathfrak{G}, H_1(C_K, \widehat{L})) \longrightarrow 0 \qquad \textbf{(A)}$$
$$\|$$
$$\widehat{H}^0(\mathfrak{G}, \widehat{L} \otimes C_K)$$

Here the top sequence is derived from the standard Lydon–Hochschild–Serre spectral sequence, the bottom sequence comes from the definition of the Tate cohomology groups and the fourth vertical arrow is given by taking cup products with the fundamental class $\overline{u_{K/F}}$.

Now since we will be trying to show this diagram commutes, it will be useful to see how the maps involved

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[5]Here (as usual) by $\langle f(g), x(g) \rangle$ we mean $f(g)(x(g))$

work.

- If $x \in Z_1(C_K, \widehat{L})$, then $\mathrm{Cor}(\overline{x})$ is in the class containing the 1-cycle $y \in Z_1(W_{K/F}, \widehat{L})$ such that $y(w) = x(w)$ if $w \in C_K$ and $y(w) = 0$ elsewhere.

- If $x \in Z_1(W_{K/F}, \widehat{L})$, then $\mathrm{Coinf}(\overline{x}) = \overline{y}$, where y is the 1-cycle in $Z_1(G, \widehat{L})$ such that

$$y(g) = \sum_{a \in C_K} x(aw_g).$$

Note that both of these maps will send cycles to cycles and boundaries to boundaries, so they are well-defined.

Our goal is to first show **(A)** commutes. Once we have this, it follows at once (by a simple diagram chase) that Tr_1 is surjective; we will then prove that Tr_1 is injective to finish the proof that Tr_1 is an isomorphism .

We first need to define Tr_1 and show that its image in in $H_1(C_K, \widehat{L})^{\mathfrak{G}}$. To do this, the strategy is to use dimension shifting and the explicit description of Tr_0 given in 1.36.

Before we begin, note that for any $g \in \mathfrak{G}$ and $w \in W_{K/F}$, we have that $w_g w \in W_{K/F}$ belongs to a unique left coset of C_K in $W_{K/F}$. Therefore there is a unique element $u(w_g, w) \in C_K$ and a unique $j(g) \in \mathfrak{G}$, such that

$$w_g w = u(w_g, w) w_{j(g)},$$

where j is just a permutation of the elements of $\mathfrak{G}$.

This can be related to the fundamental class $\overline{u_{K/F}} \in H^2(\mathfrak{G}, C_K)$, by noting that the 2-cocycle $u_{K/F}$ representing the fundamental class has the property that for each $g, g' \in \mathfrak{G}$, $w_g w_{g'} = u_{K/F}(g, g') w_{gg'}$. Therefore if $w = a w_{g'} \in W_{K/F}$ we have

$$u(w_g, w) = w_g a w_g^{-1} u_{K/F}(g, g').$$

Proposition 4.9. *If $x \in Z_1(W_{K/F}, \widehat{L})$, then we have a well-defined map*

$$\left(\mathrm{Tr}_1(x)\right)(a) = \sum_{u(w_g, w) = a} w_g x(w), \qquad \text{for all } a \in C_K.$$

Here the sum on the right is taken over all $g \in \mathfrak{G}$ and $w \in W_{K/F}$ such that $u(w_g, w) = a$. Furthermore the image of Tr_1 is in $H_1(C_K, \widehat{L})^{\mathfrak{G}}$.

Proof. We begin by considering the exact sequence

$$0 \longrightarrow I_W \longrightarrow \mathbb{Z}[W_{K/F}] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where ϵ is the augmentation map $\sum_i n_i g_i \to \sum_i n_i$. This is split over $\mathbb{Z}$, so it remains exact when tensored with $\widehat{L}$, and thus we get the exact sequence of $W_{K/F}$ modules

$$0 \longrightarrow I_W \otimes \widehat{L} \longrightarrow \mathbb{Z}[W_{K/F}] \otimes \widehat{L} \longrightarrow \mathbb{Z} \otimes \widehat{L} \longrightarrow 0.$$

Note that in this sequence we have $W_{K/F}$ acting diagonally on each of the terms, but we can find a $W_{K/F}$-module isomorphism that gives $\mathbb{Z}[W_{K/F}] \otimes \widehat{L}$ an action only on the first term of the tensor product (see [2], p. 58); this then makes $\mathbb{Z}[W_{K/F}] \otimes \widehat{L}$ is an induced module, which means that for any subgroup S of $W_{K/F}$ we have

$$H_n(S, \mathbb{Z}[W_{K/F}] \otimes \widehat{L}) = 0, \qquad \text{for } n > 0.$$

So if we now identify $\mathbb{Z} \otimes \widehat{L}$ with $\widehat{L}$ we get the exact sequence

$$0 \longrightarrow I_W \otimes \widehat{L} \longrightarrow \mathbb{Z}[W_{K/F}] \otimes \widehat{L} \longrightarrow \widehat{L} \longrightarrow 0$$

where the middle term, is an induced module, so we can use dimension shifting to get a well-defined isomorphism

$$\delta : H_1(W_{K/F}, \widehat{L}) \xrightarrow{\sim} H_0(W_{K/F}, I_W \otimes \widehat{L}),$$

that sends $\overline{z} \in H_1(W_{K/F}, \widehat{L})$ to the class of

$$\sum_{w \in W_{K/F}} (w^{-1} - 1)(1 \otimes z(w)). \qquad \text{(Here the action is diagonal)}.$$

Now by Proposition 1.36 (2), we get the following commutative diagram

$$
\begin{array}{ccc}
H_1(W_{K/F}, \widehat{L}) & \xrightarrow{\sim} & H_0(W_{K/F}, I_W \otimes \widehat{L}) \\
\downarrow{\scriptstyle \mathrm{Tr}_1} & & \downarrow{\scriptstyle \mathrm{Tr}_0} \\
H_1(C_K, \widehat{L}) & \xrightarrow{\sim} & H_0(C_K, I_W \otimes \widehat{L}),
\end{array}
$$

with the horizontal isomorphisms given by δ, defined above. Now take a 1-cycle $x \in Z_1(W_{K/F}, \widehat{L})$, under δ its image in $H_0(W_{K/F}, I_W \otimes \widehat{L})$ is in the class of

$$\sum_{w \in W_{K/F}} (w^{-1} - 1)(1 \otimes x(w)).$$

If we then apply Tr_0, we get that it maps to the class of

$$\sum_g \sum_w w_g \cdot (w^{-1} - 1)(1 \otimes x(w)) = \sum_g \sum_w (w_g w^{-1}(1 \otimes x(w)) - w_g(1 \otimes x(w)))$$

$$= \sum_g \sum_w w_g w^{-1}(1 \otimes x(w)) - \sum_g \sum_w w_g(1 \otimes x(w)) \qquad (\blacklozenge)$$

in $H_0(C_K, I_W \otimes \widehat{L})$. Now we since we can write $w_g w = u(w_g, w) w_{j(g)}$, we can use this and the fact that j is just a permutation of the elements of $\mathfrak{G}$, to write $(\blacklozenge)$ as

$$\sum_h \sum_w u(w_h, w)^{-1} w_h(1 \otimes x(w)) - \sum_g \sum_w w_g(1 \otimes x(w))$$

which after changing the summation index in the second term gives

$$\sum_h \sum_w (u(w_h, w)^{-1} - 1) w_h(1 \otimes x(w)).$$

This can be rewritten as

$$\sum_{a \in C_K} \left\{ (a^{-1} - 1) \sum_{u(w_h, w) = a} w_h(1 \otimes x(w)) \right\}.$$

Recall that we define the action of $w_g \in W_{K/F}$ on $a \otimes b$ as $w_g(a \otimes b) = w_g a \otimes w_g b$ and also note that

$$(w_h - 1) \otimes w_h x(w) = w_h(1 \otimes x(w)) - (1 \otimes w_h x(w)),$$

but the term on the left is clearly in $I_W \otimes \widehat{L}$ so by definition of H_0 we have that the sum above is in the same class as

$$\sum_{a \in C_K} \left\{ (a^{-1} - 1) \sum_{u(w_h, w) = a} 1 \otimes w_h x(w) \right\},$$

in $H_0(C_K, I_W \otimes \widehat{L})$. But this is just the image under δ of the class of the 1-cycle $y \in Z_1(C_K, \widehat{L})$, where y is defined as

$$y : a \mapsto \sum_{u(w_h, w) = a} w_h x(w).$$

Observe that this is indeed a 1-cycle, since C_K acts trivially on $\widehat{L}$, so

$$\sum_{a \in C_K} a^{-1} y(a) = \sum_{a \in C_K} y(a).$$

Furthermore it has finite support since x has finite support. So by dimension shifting, it follows that that $\mathrm{Tr}_1(\overline{x}) = \overline{y}$.

Lastly we need to show that the image of Tr_1 is in $H_1(C_K, \widehat{L})^{\mathfrak{G}}$, for which it suffices to show that for all $g \in \mathfrak{G}$ and $x \in Z_1(W_{K/F}, \widehat{L})$, the class of $g \cdot \mathrm{Tr}_1(\overline{x})$ is the same as the class of $\mathrm{Tr}_1(\overline{x})$. So fix $g \in \mathfrak{G}$ and for any $\mathfrak{G}$-module A let

$$(g \cdot) : A \longrightarrow A, \qquad a \longmapsto g \cdot a.$$

Now from the definition of Tr_0 (see 1.36 (1)) we see that the image of Tr_0 is in $H_0(C_K, I_W \otimes \widehat{L})^{\mathfrak{G}}$. Therefore we have the following commutative diagram.

$$
\begin{array}{ccc}
H_1(W_{K/F}, \widehat{L}) & \xrightarrow{\ \sim\ } & H_0(W_{K/F}, I_W \otimes \widehat{L}) \\
\downarrow{\scriptstyle \mathrm{Tr}_1} & & \downarrow{\scriptstyle \mathrm{Tr}_0} \\
H_1(C_K, \widehat{L}) & \xrightarrow{\ \sim\ } & H_0(C_K, I_W \otimes \widehat{L}) \\
\downarrow{\scriptstyle g \cdot} & & \downarrow{\scriptstyle g \cdot} \\
H_1(C_K, \widehat{L}) & \xrightarrow{\ \sim\ } & H_0(C_K, I_W \otimes \widehat{L})
\end{array}
$$

From this and Proposition 1.36 we get that the class of $(g \cdot \mathrm{Tr}_1)(\overline{x})$ is the same as the class of $\mathrm{Tr}_1(\overline{x})$ for all $x \in Z_1(W_{K/F}, \widehat{L})$. So the image of Tr_1 is in $H_1(C_K, \widehat{L})^{\mathfrak{G}}$.

$\square$

Proposition 4.10. *The square*

$$
\begin{array}{ccc}
H_1(C_K, \widehat{L}) & \xrightarrow{\ \mathrm{Cor}\ } & H_1(W_{K/F}, \widehat{L}) \\
\downarrow{\scriptstyle N_{\mathfrak{G}}} & & \downarrow{\scriptstyle \mathrm{Tr}_1} \\
0 \longrightarrow N_G(H_1(C_K, \widehat{L})) & \hookrightarrow & H_1(C_K, \widehat{L})^{\mathfrak{G}}
\end{array}
$$

is commutative.

Proof. We only need to show that $\mathrm{Tr}_1 \circ \mathrm{Cor} = N_{\mathfrak{G}}$. For this we choose x to be a 1-cycle in $Z_1(W_{K/F}, \widehat{L})$. If x comes from the corestriction of a 1-cycle in $Z_1(C_K, \widehat{L})$, then x will only have support in C_K. So we have

$$(\text{Tr}_1 \circ \text{Cor})(x(a)) = y(a) = \sum_{u(w_h, w) = a} w_h(\text{Cor}(x))(w).$$

Now since $\text{Cor}(x)$ only has support in C_K, we can write this as

$$\sum_{u(w_h, b) = a} w_h x(b),$$

where $b \in C_K$ and the sum runs over all $b \in C_K$ and all $h \in \mathfrak{G}$, such that $u(w_h, b) = a$. Now, by noting that in this case for all $h \in \mathfrak{G}$ and $b \in C_K$, we have $w_h b = u(w_h, b) w_h$; we can rewrite this as

$$\sum_{w_h b w_h^{-1} = a} w_h x(b) = \sum_{h \in G} h x(h^{-1}(a)) = \sum_{h \in G} h \cdot x(a) = N_{\mathfrak{G}}(x(a)),$$

hence we get the result.

In fact, this holds in in general for all Tr_n, see [9], Proposition (9.5), p. 82. $\qquad \square$

Proposition 4.11. *The square*

$$\begin{array}{ccc}
H_1(W_{K/F}, \widehat{L}) & \xrightarrow{\text{Coinf}} & H_1(\mathfrak{G}, \widehat{L}) \\
\downarrow{\scriptstyle \text{Tr}_1} & & \downarrow{\scriptstyle \cup \overline{u}_{K/F}} \\
H_1(C_K, \widehat{L})^{\mathfrak{G}} & \longrightarrow & \widehat{H}^0(\mathfrak{G}, \text{Hom}(L, C_K))
\end{array}$$

is commutative.

Once we have the commutativity of this square we will at once have that diagram **(A)** is commutative (after using the isomorphism $H_1(C_K, \widehat{L}) \cong \text{Hom}(L, C_K)$). Before we can prove 4.11 we first need a way to express the action of taking cup products, in terms of cycles and cocycles, for which we have the following three results.

Lemma 4.12. *Let G be a finite group and let A, B be G-modules where we make the notational convention that for $a \in A$ with $N_G(a) = 0$, we let $\overline{a}^0, \overline{a}_{-1}$ denote the canonical images of a in $\widehat{H}^0(G, A), \widehat{H}^{-1}(G, A)$ respectively.*

Let $a \in A$ with $N_G(a) = 0$ and $\overline{r}$ is the class of a 1-cocycle $r \in Z^1(G, B)$. Then

$$\overline{a}_{-1} \cup \overline{r} = \overline{c}^0,$$

where

$$c = - \sum_{g \in G} ga \otimes r(g).$$

Proof. See [10] Lemma 2, p. 176–177. $\qquad \square$

This Lemma is just what we need to be able to express the action of taking cup products in terms in cycles and cocycles. Now recall that for any finite group G and G-module B we can form the exact sequence

$$0 \longrightarrow I_G \otimes B \longrightarrow \mathbb{Z}[G] \otimes B \longrightarrow B \longrightarrow 0$$

as we did in Proposition 4.11. Similarly since the category of G-modules has enough injectives we can find G-modules B', B'' such that

$$0 \longrightarrow B \longrightarrow B' \longrightarrow B'' \longrightarrow 0$$

is an exact sequence and B' is an induced module. We can then use these exact sequences to get isomorphisms:

$$\delta : \widehat{H}^n(G, B) \longrightarrow \widehat{H}^{n+1}(G, I_G \otimes B);$$
$$\partial : \widehat{H}^n(G, B'') \longrightarrow \widehat{H}^{n+1}(G, B).$$

In particular note that when $n = -1$, the isomorphisms are given by N'_G (see 1.40). So we get

$$\delta : \widehat{H}^{-1}(G, B) \longrightarrow \widehat{H}^0(G, I_G \otimes B)$$

$$\overline{y} \longmapsto \left[\sum_{g \in G} g \otimes g.y \right] \qquad \text{(recall that } [\ ] \text{ also denotes cohomology class)}$$

and

$$\partial : \widehat{H}^{-1}(G, B'') \longrightarrow \widehat{H}^0(G, B)$$

$$\overline{x} \longmapsto \left[\sum_{g \in G} g.x \right].$$

Proposition 4.13. *Let G be a finite group, and let A, B be G-modules. If $f \in \widehat{Z}^1(G, A)$ is a 1-cocycle and $x \in \widehat{Z}^{-2}(G, B)$ is a 1-cycle, then $\overline{x} \cup \overline{f}$ is in the class of*

$$F := - \sum_{g \in G} x(g) \otimes f(g)$$

in $\widehat{H}^{-1}(G, B \otimes A)$.

Proof. First note that since we have an isomorphism $\widehat{H}^{-1}(G, B \otimes A) \cong \widehat{H}^0(G, I_G \otimes B \otimes A)$ given by N'_G, which for ease of notation we denote simply by δ. In order to prove the result, it is enough to check that the class of $\delta(\overline{x} \cup \overline{f}) = \delta(\overline{x}) \cup \overline{f}$ in $\widehat{H}^0(G, I_G \otimes B \otimes A)$ is the class containing

$$\delta(F) = - \sum_{g, h \in G} h \otimes hx(g) \otimes hf(g).$$

Now as before, we have that under δ, the image of $\overline{x}$ in $\widehat{H}^{-1}(G, I_G \otimes B)$ is in the class of

$$b = \sum_{g \in G} (g^{-1} - 1)(1 \otimes x(g)).$$

Now notice that

$$N_G(b) = \sum_{g, h \in G} (hg^{-1} - h)(1 \otimes x(g)) = \sum_{g,h} hg^{-1}(1 \otimes x(g)) - \sum_{g,h} h(1 \otimes x(g)).$$

Since we are working in $I_G \otimes B$ we have that this can be rewritten as

$$\sum_{g,h} 1 \otimes hg^{-1}x(g) - \sum_{g,h} 1 \otimes hx(g).$$

But now recall that since x is a 1-cycle, we have

$$\sum_{g \in G} g^{-1}x(g) = \sum_{g} x(g),$$

which combined with the above, tells us that $N_G(b) = 0$.

Therefore we can apply Lemma 4.12 with A and B replaced by $I_G \otimes B$ and A respectively, to get that the class of $\bar{b} \cup \bar{f}$ in $\widehat{H}^0(G, I_G \otimes B \otimes A)$ is the class containing

$$-\sum_{h \in G} h.b \otimes f(h) = -\sum_{h,g \in G} hg^{-1} \otimes hg^{-1}x(g) \otimes f(h) + \sum_{h,g \in G} h \otimes hx(g) \otimes f(h)$$

$$= -\sum_{h,g} h \otimes hx(g) \otimes f(hg) + \sum_{h,g} h \otimes hx(g) \otimes f(h).$$

However, since f is a 1-cocycle we have that $f(hg) = f(h) + hf(g)$, which after substituting gives

$$-\sum_{g,h \in G} h \otimes hx(g) \otimes hf(g).$$

$\square$

Now we can use this to get a result for 2-cocycles.

Proposition 4.14. *Let G be a finite group and let A, B be G-modules. If $f \in \widehat{Z}^2(G, B)$ is a 2-cocycle and $x \in \widehat{Z}^{-2}(G, A)$ is a 1-cycle, then the class of $\bar{f} \cup \bar{x}$ in $\widehat{H}^0(G, B \otimes A)$ is the class containing*

$$\sum_{g,h \in G} f(g,h) \otimes gx(h)$$

Note that by 1.51 (b) we have $\bar{f} \cup \bar{x} = \bar{x} \cup \bar{f}$.

Proof. (Based on J.P Serre [10], Lemma 4, p. 178.) We begin by noting that since we have an exact sequence

$$0 \longrightarrow B \longrightarrow B' \longrightarrow B'' \longrightarrow 0,$$

with B' an induced module, then $H^2(G, B') = 0$. This means we can find a 1-cochain $f' : G \longrightarrow B'$, such that

$$f(g,h) = gf'(h) - f'(gh) + f'(g).$$

Now if we compose f' with the map $B' \longrightarrow B''$, we get a 1-cocycle $f'' : G \longrightarrow B''$, such that $\partial(\bar{f''}) = \bar{f}$. We can then use this and the previous proposition to see that

$$[\bar{f} \cup \bar{x}] = [\partial(\bar{f''}) \cup \bar{x}] = N_G'([\bar{f''} \cup \bar{x}]) \stackrel{(4.13)}{=} N_G'\left(\left[\sum_{h \in G} f''(h) \otimes x(h)\right]\right) = \left[\sum_{g,h \in G} g \cdot f'(h) \otimes gx(h)\right]. \qquad (\dagger)$$

(In the last equality we change from f'' to f' since by definition of N_G' we must first lift to B'.)

Now, we know that

$$g \cdot f'(h) = f(g,h) + f'(gh) - f'(g),$$

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so (†) becomes the class containing

$$\sum_{g,h\in G}\{f(g,h)+f'(gh)-f'(g)\}\otimes gx(h)=\sum_{g,h}f(g,h)\otimes gx(h)+\sum_{g,h}(f'(gh)-f'(g))\otimes gx(h).$$

Therefore in order to finish the proof we have to show that the second term is actually zero. By changing summation indexes we have

$$\sum_{g,h}(f'(gh)-f'(g))\otimes gx(h)=\sum_{g,h}f'(gh)\otimes gx(h)-\sum_{g,h}f'(g)\otimes gx(h)=\sum_{g,h}f'(g)\otimes gh^{-1}x(h)-\sum_{g,h}f'(g)\otimes gx(h)$$

$$=\sum_{g}f'(g)\otimes g\left\{\sum_{h}(h^{-1}-1)x(h)\right\}\overset{(1.35)}{=}0.$$

$\square$

Now with this result we can prove Proposition 4.11.

Proof. We begin by taking a 1-cycle $x\in Z_1(W_{K/F},\widehat{L})$. From Proposition 4.9 we have that its image in $H_1(C_K,\widehat{L})^{\mathfrak{G}}$ under Tr_1 is in the class of the 1-cycle

$$y:a\longmapsto\sum_{u(w_h,w)=a}w_hx(w).$$

As before the sum is taken over all $h\in\mathfrak{G}$ and $w\in W_{K/F}$ such that $u(w_h,w)=a$. Now under the isomorphism of Proposition 4.5 and the natural map $H_1(C_K,\widehat{L})^{\mathfrak{G}}\longrightarrow\widehat{H}^0(\mathfrak{G},H_1(C_K,\widehat{L}))$, we get that the image $\overline{y}$ in $\widehat{H}^0(\mathfrak{G},\mathrm{Hom}(L,C_K))$ is in the class containing the homomorphism

$$\lambda\longmapsto\prod_{a\in C_K}a^{\langle\lambda,y(a)\rangle}=\prod_{a}\prod_{u(w_h,w)=a}a^{\langle\lambda,w_hx(w)\rangle}=\prod_{h,w}u(w_h,w)^{\langle\lambda,w_hx(w)\rangle}.$$

Now since each $w\in W_{K/F}$ can be written as aw_g for some $g\in\mathfrak{G}$ and $a\in C_K$, we have

$$u(w_h,w)=w_haw_h^{-1}u_{K/F}(h,g).$$

So we can rewrite the homomorphism above as

$$\lambda\longmapsto\left\{\prod_{g,h,a}(w_haw_h^{-1})^{\langle\lambda,w_hx(aw_g)\rangle}\right\}\left\{\prod_{g,h,a}u_{K/F}(h,g)^{\langle\lambda,w_hx(aw_g)\rangle}\right\}.$$

Now notice that the first term is the image of a norm, so since we are working in the zeroth Tate cohomology group, we get that this homomorphism is in the same class as

$$\beta:\lambda\longmapsto\prod_{g,h,a}u_{K/F}(h,g)^{\langle\lambda,w_hx(aw_g)\rangle}=\prod_{g,h}u_{K/F}(h,g)^{\langle\lambda,hz(g)\rangle},$$

where

$$z(g)=\sum_{a\in C_K}x(aw_g),\qquad\text{for all }g\in\mathfrak{G}.$$

Alternatively if we take $x\in Z_1(W_{K/F},\widehat{L})$ and go along the top of the square we have that $\mathrm{Coinf}(\overline{x})=\overline{z}$. So in order to show the square commutes we must show that $[\overline{z}\cup\overline{u_{K/F}}]=\overline{\beta}$.

Now let $B=C_K$ and $A=\widehat{L}$ in Proposition 4.14, then we can take $f(h,g)=u_{K/F}(h,g)$ so that $\overline{u_{K/F}}$ is the

fundamental class, and we take x to be z. Then Proposition 4.14 tell us that the class of $\overline{z} \cup \overline{u_{K/F}}$ in $\widehat{H}^0(\mathfrak{G}, \widehat{L} \otimes C_K)$ is

$$\sum_{g,h \in \mathfrak{G}} hz(g) \otimes u_{K/F}(h, g),$$

which maps to the homomorphism

$$\beta : \lambda \longmapsto \prod_{g,h \in \mathfrak{G}} u_{K/F}(h, g)^{\langle \lambda, hz(g) \rangle}$$

in $\widehat{H}^0(\mathfrak{G}, \operatorname{Hom}(L, C_K))$ as required.

$\square$

Thus we have that the second square in diagram **(A)** commutes. As we mentioned before, this now tells us that Tr_1 is surjective (this is just a simple diagram chase). Now we are left proving that Tr_1 is injective. For this note that, once again, that from the commutativity of **(A)**, its enough to show that the kernel of the map

$$\mathrm{Cor} : H_1(C_K, \widehat{L}) \longrightarrow H_1(W_{K/F}, \widehat{L})$$

is equal to the kernel of the map

$$N_{\mathfrak{G}} : H_1(C_K, L) \longrightarrow H_1(C_K, \widehat{L}).$$

In other words, we want to show that the kernel of the corestriction consists precisely, of the elements of norm zero.

This is equivalent to showing that the the image of

$$\mathrm{Cor}' : \operatorname{Hom}(H_1(W_{K/F}, \widehat{L}), \mathbb{Q}/\mathbb{Z}) \longrightarrow \operatorname{Hom}(H_1(C_K, \widehat{L}), \mathbb{Q}/\mathbb{Z}),$$

consists of homomorphisms that vanish on elements of norm zero[6]. Now using Proposition 4.7, we have isomorphisms

$$F' : \operatorname{Hom}(H_1(W_{K/F}, \widehat{L}), \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(W_{K/F}, \operatorname{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}))$$
$$F : \operatorname{Hom}(H_1(C_K, \widehat{L}), \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(C_K, \operatorname{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z})).$$

So, its enough to show that the image of

$$H^1(W_{K/F}, \operatorname{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z})) \longrightarrow H^1(C_K, \operatorname{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z})),$$

consists of elements corresponding (under F) to homomorphisms that vanish on elements of norm zero. In other words, we want to show that given $\overline{\psi} \in H^1(C_K, \operatorname{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}))$, we can extend this to a $\overline{\Psi} \in H^1(W_{K/F}, \operatorname{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}))$ if and only if $\overline{\psi} = F(\varphi)$, where φ is a homomorphism vanishes on elements of norm zero.

Now following Langlands [1] p.13, we reformulate this problem as follows: Recall from Proposition 4.2 we have the following exacts sequence

$$0 \longrightarrow C_K \longrightarrow W_{K/F} \xrightarrow{\sigma} \operatorname{Gal}(K/F) \longrightarrow 0,$$

whose class in $H^2(\mathfrak{G}, C_K)$ is $\overline{u_{K/F}}$ (the fundamental class). Also, we can use the action of $\mathfrak{G}$ on $\widehat{L}$, to give $\operatorname{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z})$ a $\mathfrak{G}$-action, by letting $\mathfrak{G}$ act trivially on $\mathbb{Q}/\mathbb{Z}$, and we can form the semi-direct product

$$\operatorname{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}) \rtimes \mathfrak{G}.$$

---

[6]Note that $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is an exact contravariant functor since $\mathbb{Q}/\mathbb{Z}$ is divisible

Now suppose we had the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_K & \longrightarrow & W_{K/F} & \overset{\sigma}{\longrightarrow} & \mathfrak{G} & \longrightarrow & 0 \\
& & \downarrow{\psi} & & \downarrow{\Psi} & & \downarrow{id} & & \\
0 & \longrightarrow & \mathrm{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}) \rtimes \mathfrak{G} & \longrightarrow & \mathfrak{G} & \longrightarrow & 0
\end{array}
$$

We can define a 1-cochain f by $\Psi(w) = f(w) \times \sigma(w)$, and in fact we see that $f \in Z^1(W_{K/F}, \mathrm{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}))$. Conversely given $f \in Z^1(W_{K/F}, \mathrm{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}))$ we can define $\Psi = f \times \sigma$, such that Ψ together with its restriction ψ to C_K will make the above diagram commute. Now what we want to prove is that given a homomorphism

$$
\psi : C_K \longrightarrow \mathrm{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}),
$$

we can extend this to a homomorphism

$$
\Psi : W_{K/F} \longrightarrow \mathrm{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}) \rtimes \mathfrak{G}
$$

making the diagram commute if and only if ψ corresponds (under F) to a homomorphism $\varphi : H_1(C_K, \widehat{L}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ that vanishes on elements of norm zero.

Now for this we have the following result:

Lemma 4.15. *Let the following short exact sequences of groups*

$$
0 \longrightarrow A \longrightarrow U \longrightarrow G \longrightarrow 0 \qquad and \qquad 0 \longrightarrow A' \longrightarrow U' \longrightarrow G' \longrightarrow 0
$$

represent two group extensions, where A, A' are abelian. Then, if we are given homomorphisms $f : A \longrightarrow A'$ and $h : G \longrightarrow G'$, there exists a homomorphism $F : U \longrightarrow U'$ such that the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & U & \longrightarrow & G & \longrightarrow & 0 \\
& & \downarrow{f} & & \downarrow{F} & & \downarrow{h} & & \\
0 & \longrightarrow & A' & \longrightarrow & U' & \longrightarrow & G' & \longrightarrow & 0
\end{array}
$$

is commutative, if and only if:

1. *f is a G-homomorphism, G acting on A' through h.*

2. *$f_* \alpha = h^* \alpha'$, where $\alpha \in H^2(G, A)$ and $\alpha' \in H^2(G', A')$ represent the extensions of U and U' respectively. (Here f_* and h^* represent the homomorphisms induced by $(1, f) : (G, A) \longrightarrow (G, A')$ and $(h, 1) : (G, A') \longrightarrow (G', A')$ respectively.)*

Proof. See [13], Theorem 2, p. 130. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Using this result we have that ψ will extend to Ψ if and only is ψ is G-invariant and[7] $\psi_*(\overline{u_{K/F}}) = 0$.

Now if

$$
\psi : C_K \longrightarrow \mathrm{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}),
$$

corresponds (under F) to

$$
\varphi : H_1(C_K, \widehat{L}) \longrightarrow \mathbb{Q}/\mathbb{Z},
$$

---

[7]This is because the class representing a semi-direct product is zero

then it is easy to see that for all $a \in C_K$ and $\widehat{\lambda} \in \widehat{L}$, we have $(\psi(a))(\widehat{\lambda}) = \varphi(\widehat{\lambda} \otimes a)$, where we are using the fact that $H_1(C_K, \widehat{L}) \cong \widehat{L} \otimes C_K$.

Now to prove (1) of the above Lemma, we want to show that ψ is $\mathfrak{G}$-invariant if φ vanishes on elements of norm zero. But by the above, we see that ψ is $\mathfrak{G}$-invariant, if and only if for all $\widehat{\lambda} \in \widehat{L}$, $a \in C_K$ and all $g \in \mathfrak{G}$, we have

$$\psi(g \cdot a)(\widehat{\lambda}) = \psi(a)(g^{-1}\widehat{\lambda}),$$

which is true if and only if

$$\varphi(\widehat{\lambda} \otimes ga) = \varphi(g^{-1}\widehat{\lambda} \otimes a).$$

But this is true if for all $a \otimes \widehat{\lambda} \in C_K \otimes \widehat{L}$, we have

$$g \cdot (\widehat{\lambda} \otimes a) - (\widehat{\lambda} \otimes a) \in \ker(\varphi),$$

which is true if φ vanishes on elements of norm zero.

So it remains to prove that $\psi_*(\overline{u_{K/F}}) = 0$.

For each pair of homomorphisms $\psi : C_K \longrightarrow \text{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z})$ and $\varphi : H_1(C_K, \widehat{L}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $\psi = F(\varphi)$, we have the following commutative diagram

$$
\begin{array}{ccc}
\widehat{L} \otimes C_K & \xrightarrow{\ id \otimes \psi\ } & \widehat{L} \otimes \text{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}) \\
\downarrow & & \downarrow \\
H_1(C_K, \widehat{L}) & \xrightarrow{\quad \varphi \quad} & \mathbb{Q}/\mathbb{Z}
\end{array}
$$

where the first vertical arrow is the isomorphism (see 4.5) that sends $\widehat{\lambda} \otimes a$ to the 1-cycle that is zero except at a where it is $\widehat{\lambda}$, for $\widehat{\lambda} \in \widehat{L}$ and $a \in C_K$.

Now if ψ (and hence φ) is $\mathfrak{G}$-invariant we get the following commutative diagram:

$$
\begin{array}{ccc}
\widehat{H}^{-3}(\mathfrak{G}, \widehat{L}) \otimes \widehat{H}^2(\mathfrak{G}, C_K) & \xrightarrow{\ id \otimes \psi_*\ } & \widehat{H}^{-3}(\mathfrak{G}, \widehat{L}) \otimes \widehat{H}^2(\mathfrak{G}, \text{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z})) \\
\downarrow{\mu} & & \downarrow{\nu} \\
\widehat{H}^{-1}(\mathfrak{G}, H_1(C_K, \widehat{L})) & \xrightarrow{\quad \varphi' \quad} & \widehat{H}^{-1}(\mathfrak{G}, \mathbb{Q}/\mathbb{Z}),
\end{array}
$$

where the vertical arrows are given by taking cup products, the map φ' is induced by φ and ψ_*, is the map induced by ψ with the notation of 4.15. Note that for $\gamma \in \widehat{H}^{-3}(\mathfrak{G}, \widehat{L})$, the map

$$\gamma \longrightarrow \mu(\gamma \otimes \overline{u_{K/F}}),$$

gives an isomorphism

$$\widehat{H}^{-3}(\mathfrak{G}, \widehat{L}) \longrightarrow \widehat{H}^{-1}(\mathfrak{G}, H_1(C_K, \widehat{L})).$$

This is the isomorphism given by taking cup-product with the fundamental class $\overline{u_{K/F}}$ (see 1.52).

Similarly have that for $\beta \in \widehat{H}^{-3}(\mathfrak{G}, \widehat{L})$, and $\gamma \in \widehat{H}^2(\mathfrak{G}, \text{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}))$ the map

$$\beta \longrightarrow \nu(\beta \otimes \gamma)$$

is an isomorphism for all $\gamma \in \widehat{H}^2(\mathfrak{G}, \text{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}))$. For a proof of this see [9], Corollary 7.3, p. 146 and note that in this case, both $\widehat{H}^{-3}(\mathfrak{G}, \widehat{L})$ and $\widehat{H}^2(\mathfrak{G}, \text{Hom}(\widehat{L}, \mathbb{Q}/\mathbb{Z}))$ are finite groups (see 4.18).

So, from the diagram, we see that $\psi_*(\overline{u_{K/F}}) = 0$ if and only if for all $\beta \in \widehat{H}^{-3}(\mathfrak{G}, \widehat{L})$

$$\nu(\beta \otimes \psi_*(\overline{u_{K/F}})) = 0.$$

Now going around the diagram in the other direction we see that, since the map $\gamma \longrightarrow \mu(\gamma \otimes \overline{u_{K/F}})$, is an isomorphism; we have $\psi_*(\overline{u_{K/F}}) = 0$ if and only if φ' is the zero map. This, by definition of φ and $\widehat{H}^{-1}$, is true if and only if φ vanishes on elements of norm zero.

This now completes the proof that Tr_1 is an isomorphism. Our goal now is to prove the following:

Proposition 4.16. *If $x \in Z^1(W_{K/F}, \widehat{T}_D)$, then $\Psi(\overline{x}) \in \mathrm{Hom}_{cts}(\mathrm{Hom}_{\mathfrak{G}}(L, C_K), D)$ if and only if x is a continuous 1-cocycle.*

Note here that by using Proposition 4.5, we can give $H_1(C_K, \widehat{L})$ a topology by using the natural topology on $\mathrm{Hom}(L, C_K)$, and consequently we get a topology on $H_1(W_{K/F}, \widehat{L})$ by using the fact that Tr_1 is an isomorphism.

Now in order to prove 4.16 we begin by proving the following:

Proposition 4.17. *A homomorphism $\alpha : \mathrm{Hom}_{\mathfrak{G}}(L, C_K) \longrightarrow D$ is continuous if and only if $\alpha \circ N_{\mathfrak{G}}$ is continuous.*

In order to prove this, it suffices to prove that $N_{\mathfrak{G}}(\mathrm{Hom}(L, C_K))$ (the image under $N_{\mathfrak{G}}$) is an open subgroup of $\mathrm{Hom}_{\mathfrak{G}}(L, C_K)$, since a homomorphism of topological groups is continuous if and only if it is continuous at the identity[8]. It follows that a homomorphism will be continuous on $\mathrm{Hom}_{\mathfrak{G}}(L, C_K)$ if and only if it continuous on an open subgroup.

Proposition 4.18. *For all $i \in \mathbb{Z}$, $\widehat{H}^i(\mathfrak{G}, \mathrm{Hom}(L, C_K))$ is a finite group.*

Proof. Since we have a $\mathfrak{G}$-module isomorphism between $\mathrm{Hom}(L, C_K)$ and $\widehat{L} \otimes C_K$, it is enough to prove that $\widehat{H}^i(\mathfrak{G}, \widehat{L} \otimes C_K)$ is a finite group. Now from the Tate–Nakayama Lemma we have that for all $i \in \mathbb{Z}$

$$\widehat{H}^i(\mathfrak{G}, \widehat{L}) \cong \widehat{H}^{i+2}(\mathfrak{G}, \widehat{L} \otimes C_K).$$

So we can reduce the problem to showing that $\widehat{H}^i(\mathfrak{G}, \widehat{L})$ is finite, for which we have the following result.

Lemma 4.19. *If M is a finitely generated G-module and G is a finite group, then $\widehat{H}^i(G, M)$ is a finite group.*

Proof. See [7] Proposition (3-1-9), p. 89. $\qquad\qquad\square$

Now since L is a finitely generated $\mathbb{Z}$-module, $\widehat{L}$ will also be a finitely generated $\mathbb{Z}$-module and consequently $\widehat{L}$ will also be a finitely generated $\mathbb{Z}[\mathfrak{G}]$-module, so the above Lemma applies, giving the result of Proposition 4.18.

$\qquad\qquad\square$

Applying Proposition 4.18 with $i = 0$, gives that $N_{\mathfrak{G}}(\mathrm{Hom}(L, C_K))$ has finite index in $\mathrm{Hom}_{\mathfrak{G}}(L, C_K)$. Therefore in order to prove Proposition 4.17 it suffices to prove that $N_{\mathfrak{G}}(\mathrm{Hom}(L, C_K))$ is closed in $\mathrm{Hom}_{\mathfrak{G}}(L, C_K)$, since any closed subgroup of finite index is automatically open (see Proposition 3.2 (b)).

Now observe that if K is a local field or a global function field, then we have a natural homomorphism from C_K into $\mathbb{Z}$, whose kernel U_K is known to be compact. Similarly if K is a number field, then there is a natural map from C_K to $\mathbb{R}^{>0}$, whose kernel we once again denote by U_K and is also compact (see [12] Theorem (1.6), p. 362).

With this we can form the exact sequence of abelian groups

$$1 \longrightarrow U_K \longrightarrow C_K \longrightarrow M_K \longrightarrow 1, \tag{$*$}$$

---

[8]Since once we know its continuous at the identity we can use the fact that multiplication and inversion are continuous maps to show that the homomorphism is continuous at all other points

where we set $M_K = \mathbb{Z}$ or $M_K = \mathbb{R}^{>0} \cong \mathbb{R}$ accordingly, and we call the two cases the "$\mathbb{Z}$-Case" and "$\mathbb{R}$-Case" respectively.

We start with the $\mathbb{Z}$-**Case**. Since L is a free $\mathbb{Z}$-module (hence projective) we can use $(*)$ to form the exact sequence

$$0 \longrightarrow \operatorname{Hom}(L, U_K) \xrightarrow{\lambda} \operatorname{Hom}(L, C_K) \xrightarrow{\mu} \operatorname{Hom}(L, M_K) \longrightarrow 0,$$

where we think of this as a sequence of $\mathfrak{G}$-modules by giving M_K the trivial action. Note that in the $\mathbb{Z}$-case we have

$$\widehat{H}^i(\mathfrak{G}, \operatorname{Hom}(L, M_K)) = \widehat{H}^i(\mathfrak{G}, \widehat{L}), \qquad \text{for all } i \in \mathbb{Z}$$

and Lemma 4.19 tells us that all of these groups are finite.

Proposition 4.20. *There is an injective map*

$$\psi : \left(N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K)) \cap \operatorname{Hom}(L, U_K)\right) / N_{\mathfrak{G}}(\operatorname{Hom}(L, U_K)) \longrightarrow \widehat{H}^{-1}(\mathfrak{G}, \operatorname{Hom}(L, M_K)) / \mu \widehat{H}^{-1}(\mathfrak{G}, \operatorname{Hom}(L, C_K)).$$

In order to ease notation in the proof, we set

$$B = N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K)) \cap \operatorname{Hom}(L, U_K), \qquad V = \widehat{H}^{-1}(\mathfrak{G}, \operatorname{Hom}(L, M_K)) / \mu \widehat{H}^{-1}(\mathfrak{G}, \operatorname{Hom}(L, C_K)).$$

Note that once we have this result, it will follow at once that $N_{\mathfrak{G}}(\operatorname{Hom}(L, U_K))$ has finite index in B, since V is finite by the comment above.

Proof. We begin by taking $x \in \operatorname{Hom}(L, C_K)$ such that $z = N_{\mathfrak{G}}(x) \in \operatorname{Hom}(L, U_K)$. If we let $y = \mu(x)$ with μ as above, then by exactness and the fact that μ is a homomorphism we have

$$N_{\mathfrak{G}}(y) = N_{\mathfrak{G}}(\mu(x)) = \mu(N_{\mathfrak{G}}(x)) = 0.$$

Now we claim there is a well-defined map ψ, such that $\psi(z) = \overline{y}$ where $\overline{y}$ is the image of y in V. Note that if $x \in \operatorname{Hom}(L, U_K)$, then $\overline{y}$ will be zero. To prove this claim, note that the image of y will be independent of x since on the right we quotient out by $\mu \widehat{H}^{-1}(\mathfrak{G}, \operatorname{Hom}(L, C_K))$; which means that if we had

$$z = N_{\mathfrak{G}}(x) = N_{\mathfrak{G}}(x'),$$

then letting $x - x' = r$, we would have $N_{\mathfrak{G}}(r) = 0$ and

$$y = \mu(x) = \mu(x') + \mu(r) = y' + \mu(r).$$

So when we look at $\overline{y}$ and $\overline{y}'$ we can clearly see they will represent the same element in V, hence ψ is well-defined.

In order to show the ψ is injective, it suffices to show that if $\psi(z) = 0$ for $z = N_{\mathfrak{G}}(x)$, and $x \in \operatorname{Hom}(L, C_K)$, then we can chose an $x' \in \operatorname{Hom}(L, U_K)$ such that $N_{\mathfrak{G}}(x) = N_{\mathfrak{G}}(x')$.

So suppose that

$$\psi(N_{\mathfrak{G}}(x)) = \psi(z) = 0,$$

then we can choose x such that we have $\mu(x) = y \in I_{\mathfrak{G}}(\operatorname{Hom}(L, M_K))$, hence

$$y = \sum_g (g^{-1} - 1) v_g, \qquad \text{for some } v_g \in \operatorname{Hom}(L, M_K).$$

Now since μ is surjective we can pick $u_g \in \operatorname{Hom}(L, C_K)$, such that $\mu(u_g) = v_g$ and we can also pick $x \in \operatorname{Hom}(L, C_k)$

such that $\mu(x) = y$. Now consider

$$x' = x - \sum_g (g^{-1} - 1)u_g;$$

it must lie in $\operatorname{Hom}(L, U_K)$ since $\mu(x') = \mu(x) - \sum_g (g^{-1} - 1)v_g = 0$, but we also have $N_{\mathfrak{G}}(x) = N_{\mathfrak{G}}(x')$, so we are done. $\qquad\square$

With this result we can now show that in the $\mathbb{Z}$-case, $N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K))$ is closed in $\operatorname{Hom}_{\mathfrak{G}}(L, C_K)$. First note that $L \cong \mathbb{Z}^n$ as abelian groups (for some $n \in \mathbb{Z}_{\geq 0}$), and since U_K is compact and Hausdorff, then

$$\operatorname{Hom}(L, U_K) \cong (U_K)^n$$

is also compact and Hausdorff (being the direct sum of compact and Hausdorff groups).

Also since $N_{\mathfrak{G}}$ is a continuous map, we have that $N_{\mathfrak{G}}(\operatorname{Hom}(L, U_K))$ must be closed in $\operatorname{Hom}(L, U_K)$ (being a compact subgroup of a Hausdorff group). Therefore since B is a subgroup of $\operatorname{Hom}(L, U_K)$, and $N_{\mathfrak{G}}(\operatorname{Hom}(L, U_K))$ is closed in $\operatorname{Hom}(L, U_K)$, we must have that $N_{\mathfrak{G}}(\operatorname{Hom}(L, U_K))$ is also be closed in B. We also know $N_{\mathfrak{G}}(\operatorname{Hom}(L, U_K))$ is of finite index in B. Therefore we can write

$$B = \bigcup_{k=1}^n b_k N_{\mathfrak{G}}(\operatorname{Hom}(L, U_K)), \qquad \text{for some } b_k \in B,$$

and it follows that B is closed in $\operatorname{Hom}(L, U_K)$ by 3.2 (a). Now recall that by Proposition 3.3, a normal subgroup H of a topological group G, is open if and only if the quotient topology on G/H is discrete. So since $M_K = \mathbb{Z}$ we must have that U_K is an open subgroup of C_K since we know that U_K is closed in C_K and $C_K/U_K \cong \mathbb{Z}$. Therefore $\operatorname{Hom}_{\mathfrak{G}}(L, U_K)$ is an open subgroup of $\operatorname{Hom}_{\mathfrak{G}}(L, C_K)$.

Now to finish the proof that $N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K))$ is closed in $\operatorname{Hom}_{\mathfrak{G}}(L, C_K)$ we can use Lemma 3.4, by letting

$$A = \operatorname{Hom}_{\mathfrak{G}}(L, C_K), \qquad H = \operatorname{Hom}_{\mathfrak{G}}(L, U_K), \qquad S = N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K)),$$

and noting that we are in the situation of Lemma 3.4 since

$$B = N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K)) \cap \operatorname{Hom}(L, U_K) = N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K)) \cap \operatorname{Hom}_{\mathfrak{G}}(L, U_K),$$

is closed in H. Hence in the $\mathbb{Z}$-case we have that $N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K))$ is closed in $\operatorname{Hom}_{\mathfrak{G}}(L, C_K)$.

Now we move to the $\mathbb{R}$-case. Here we are in a slightly easier situation, since in this case the exact sequence

$$1 \longrightarrow U_K \longrightarrow C_K \longrightarrow \mathbb{R}^{>0} \longrightarrow 1$$

splits as a sequence of $\mathfrak{G}$-modules. Therefore the sequence

$$0 \longrightarrow \operatorname{Hom}(L, U_K) \xrightarrow{\lambda} \operatorname{Hom}(L, C_K) \xrightarrow{\mu} \operatorname{Hom}(L, M_K) \longrightarrow 0$$

also splits as a sequence of $\mathfrak{G}$-modules, so we get

$$\operatorname{Hom}(L, C_K) = \operatorname{Hom}(L, U_K) \times \operatorname{Hom}(L, \mathbb{R}), \tag{$\ddagger$}$$

and

$$N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K)) = N_{\mathfrak{G}}(\operatorname{Hom}(L, U_K)) \times N_{\mathfrak{G}}(\operatorname{Hom}(L, \mathbb{R})).$$

Furthermore if we look at the zeroth cohomology groups of (‡) , we get

$$\mathrm{Hom}_{\mathfrak{G}}(L, C_K) = \mathrm{Hom}_{\mathfrak{G}}(L, U_K) \times \mathrm{Hom}_{\mathfrak{G}}(L, \mathbb{R}).$$

Proposition 4.21. *In the $\mathbb{R}$-case we have that $\widehat{H}^0(\mathfrak{G}, \mathrm{Hom}(L, M_K)) = 0$.*

Proof. First note that

$$\widehat{H}^0(\mathfrak{G}, \mathrm{Hom}(L, M_K)) = \widehat{H}^0(\mathfrak{G}, \mathrm{Hom}(L, \mathbb{R})) = \mathrm{Hom}_{\mathfrak{G}}(L, \mathbb{R})/N_{\mathfrak{G}}(\mathrm{Hom}(L, \mathbb{R}))$$

so the result will follow if we can show that any $\mathfrak{G}$-linear homomorphism from L to $\mathbb{R}$ can be written as the norm of some homomorphism from L to $\mathbb{R}$. Now since $\mathfrak{G}$ acts trivially on $\mathbb{R}$, we see that for any $\varphi \in \mathrm{Hom}_{\mathfrak{G}}(L, \mathbb{R})$ we have

$$N_{\mathfrak{G}}(\varphi) = m\varphi, \qquad m = |\mathfrak{G}|.$$

Therefore since $\mathbb{R}$ is a divisible group we have that $\theta = (1/m)\varphi$ is also a homomorphism from L to $\mathbb{R}$, and thus $\varphi = N_{\mathfrak{G}}(\theta)$, which gives the result. $\qquad\square$

Now the proposition above tells us that

$$N_{\mathfrak{G}}(\mathrm{Hom}(L, \mathbb{R})) = \mathrm{Hom}_{\mathfrak{G}}(L, \mathbb{R}),$$

and as before, we have that $N_{\mathfrak{G}}(\mathrm{Hom}(L, U_K))$ is closed in $\mathrm{Hom}_{\mathfrak{G}}(L, U_K)$. It then follows that

$$N_{\mathfrak{G}}(\mathrm{Hom}(L, U_K)) \times N_{\mathfrak{G}}(\mathrm{Hom}(L, \mathbb{R}))$$

is closed in

$$\mathrm{Hom}_{\mathfrak{G}}(L, U_K) \times \mathrm{Hom}_{\mathfrak{G}}(L, \mathbb{R}).$$

Hence $N_{\mathfrak{G}}(\mathrm{Hom}(L, C_K))$ is closed in $\mathrm{Hom}_{\mathfrak{G}}(L, C_K)$.

So we have shown that in both the $\mathbb{Z}$-case and $\mathbb{R}$-case $N_{\mathfrak{G}}(\mathrm{Hom}(L, C_K))$ is closed in $\mathrm{Hom}_{\mathfrak{G}}(L, C_K)$ and of finite index. Thus $N_{\mathfrak{G}}(\mathrm{Hom}(L, C_K))$ is open in $\mathrm{Hom}_{\mathfrak{G}}(L, C_K)$ by 3.2 (b), which proves Proposition 4.17.

Proposition 4.22. *A 1-cocycle in $Z^1(W_{K/F}, \widehat{T}_D)$ will be continuous if and only if its restriction to C_K is continuous.*

Proof. Clearly if $x \in Z^1_{cts}(W_{K/F}, \widehat{T}_D)$, then its restriction to C_K will also be continuous, so we only need to prove the other direction.

If $x \in Z^1(W_{K/F}, \widehat{T}_D)$ is continuous on C_K, then define $\sigma(a) = x(wa)$, where $w \in W_{K/F}$, $a \in C_K$, then in order to prove that x is continuous on the coset wC_K we only need to prove that σ is continuous. Now since x is a 1-cocycle, we have

$$\sigma(a) = x(wa) = wx(a) + x(w).$$

So as a goes through C_K, we have that $wx(-)$ is continuous since $x(-)$ is continuous on C_K and the action of $W_{K/F}$ is continuous (since it is induced from the continuous action of $\mathfrak{G}$ on $\widehat{T}_D$). Therefore since $x(w)$ is just a constant, we have that σ is continuous, and hence x is continuous on wC_K. From which it follows that x is continuous on all of $W_{K/F}$. $\qquad\square$

Now observe that we have the following diagram:

$$
\begin{array}{ccc}
H^1(W_{K/F}, \widehat{T}_D) & \stackrel{\sim}{\longrightarrow} & \mathrm{Hom}(H_1(W_{K/F}, \widehat{L}), D) \\
\downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Cor}'} \\
H^1(C_K, \widehat{T}_D) & \stackrel{\sim}{\longrightarrow} & \mathrm{Hom}(H_1(C_K, \widehat{L}), D),
\end{array}
$$

where the horizontal arrows are isomorphisms given by Proposition 4.7, Res is the standard restriction map on cohomology groups, and Cor' is the surjective map induced from

$$
\mathrm{Cor} : H_1(C_K, \widehat{L}) \longrightarrow H_1(W_{K/F}, \widehat{L}).
$$

Now it is easy to show this diagram is in fact commutative since if we begin by taking a 1-cocycle $f \in Z^1(W_{K/F}, \widehat{T}_D)$, and first move along the top of the diagram, then by remark 4.8, and the definition of Cor for homology, we get that f maps to the homomorphism sending $x \in H_1(C_K, \widehat{L})$ to

$$
\sum_{a \in C_K} \langle f(a), x(a) \rangle.
$$

But this is clearly the same as going around the diagram in the other direction.

This diagram together with Proposition 4.22, reduce Proposition 4.16 to proving the following:

Proposition 4.23. *Let*
$$
\Psi : H^1(C_K, \widehat{T}_D) \stackrel{\sim}{\longrightarrow} \mathrm{Hom}(H_1(C_K, \widehat{L}), D).
$$

If f is a 1-cocycle in $Z^1(C_K, \widehat{T}_D)$, then f is continuous if and only if $\Psi(\overline{f})$ is a continuous homomorphism in $\mathrm{Hom}(H_1(C_K, \widehat{L}), D)$ or, what is the same, in $\mathrm{Hom}(\widehat{L} \otimes C_K, D)$.

Proof. In this case we can see what f maps to. It will correspond to the homomorphism $\Psi(\overline{f}) \in \mathrm{Hom}(\widehat{L} \otimes C_K, D)$

$$
\Psi(\overline{f}) : \widehat{\lambda} \otimes a \longmapsto \langle \widehat{\lambda}, f(a) \rangle
$$

where, in this case, we have that $\langle -, - \rangle$ is the natural bilinear mapping

$$
\langle -, - \rangle : \widehat{L} \times \widehat{T}_D \longrightarrow D.
$$

This bilinear map can easily be seen to be continuous by observing that $\widehat{L}$ has the discrete topology and $\widehat{T}_D$ has topology induced by that of D. Then with this it is clear that f will be a continuous 1-cocycle if and only if $\Psi(\overline{f})$ is a continuous homomorphism. $\qquad \square$

Hence we have proven Proposition 4.16, which completes the proof of Theorem 4.4.

4.4 Application to Algebraic Tori

Our next goal is to use this to say in a bit more detail how Theorem 4.4 relates to algebraic tori.

Recall that at the start we identified $T(K)$ (the group of K-rational points) with $\mathrm{Hom}(L, K^\times)$. In the case that K is a global field, we have an exact sequence

$$
1 \longrightarrow K^\times \longrightarrow \mathbb{A}_K^\times \longrightarrow C_K \longrightarrow 1,
$$

which, since L is free (and hence projective), gives us the exact sequence

$$0 \longrightarrow \mathrm{Hom}(L, K^\times) \longrightarrow \mathrm{Hom}(L, \mathbb{A}_K^\times) \longrightarrow \mathrm{Hom}(L, C_K) \longrightarrow 0.$$
$$\| \qquad\qquad\qquad \|$$
$$T(K) \qquad\qquad\quad T(\mathbb{A}_K)$$

Hence we can identify $T(\mathbb{A}_K)/T(K)$ with $\mathrm{Hom}(L, C_K)$. Also from this we obtain a long exact sequence

$$0 \longrightarrow T(F) \longrightarrow T(\mathbb{A}_F) \longrightarrow \mathrm{Hom}_{\mathfrak{G}}(L, C_K) \longrightarrow H^1(\mathfrak{G}, T(K)) \longrightarrow \cdots,$$

which we will use later. Next we need a result that allows us to switch between $W_{K/F}$ and W_F.

Proposition 4.24. *Let D be a Hausdorff divisible abelian topological group. Then the inflation map*

$$\mathrm{Inf} : H^1_{cts}(W_{K/F}, \widehat{T}_D) \longrightarrow H^1_{cts}(W_F, \widehat{T}_D)$$

is bijective.

Proof. (See [11], p. 111) First recall that the inflation map is always injective, and so is its restriction to H^1_{cts}, so we only need to prove that in this case it is also surjective. So take a continuous 1-cocycle $f : W_F \longrightarrow \widehat{T}_D$, this will restrict to a continuous 1-cocycle in $Z^1_{cts}(W_K, \widehat{T}_D)$. Now the kernel of f must contain the commutator group of W_K since $\widehat{T}_D$ is commutative. Furthermore since $\widehat{T}_D$ is Hausdorff, the kernel of f must also be closed. So f must be trivial on W_K^c and hence it must factor through $W_F/W_K^c = W_{K/F}$, giving that Inf is also surjective. $\qquad\square$

Now, if F is a global field, we say an element in $H^1_{cts}(W_F, \widehat{T}_D)$ is locally trivial if it restricts to zero in $H^1_{cts}(W_{F_v}, \widehat{T}_D)$ for all places v of F.

Theorem 4.25. *(a) If K is a local field, then $H^1_{cts}(W_{K/F}, \widehat{T}_D)$ is isomorphic to $\mathrm{Hom}_{cts}(T(F), D)$*

(b) Let D' be a divisible abelian topological group such that for any finite group X, $\mathrm{Hom}(X, D')$ is finite, and let

$$\widehat{T}_{D'} = \mathrm{Hom}(\widehat{L}, D').$$

If K is a global field, then there is a canonical surjective homomorphism

$$\widehat{H}^1_{cts}(W_{K/F}, \widehat{T}_{D'}) \longrightarrow \mathrm{Hom}_{cts}(T(\mathbb{A}_F)/T(F), D'),$$

and the kernel of this homomorphism is finite. Furthermore, if D' is also Hausdorff, then the kernel consists of the locally trivial classes.

Proof. (a) This follows immediately from Theorem 4.4, since in this case $C_K = K^\times$ and therefore $T(F) = T(K)^{\mathfrak{G}} = \mathrm{Hom}_{\mathfrak{G}}(L, C_K)$, and the isomorphism in question is exactly the one we found in Theorem 4.4.

(b) We begin by considering the following commutative diagram

$$0 \longrightarrow T(K) \longrightarrow T(\mathbb{A}_K) \longrightarrow \operatorname{Hom}(L, C_K) \longrightarrow 0.$$

$$0 \longrightarrow T(F) \longrightarrow T(\mathbb{A}_F) \longrightarrow \operatorname{Hom}_{\mathfrak{G}}(L, C_K) \longrightarrow \cdots$$

(with vertical maps $N_{\mathfrak{G}}$)

From before, we already know both rows are exact, and the commutativity of the diagram tells us that we must have $N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K))$ contained in $T(\mathbb{A}_F)/T(F)$. Now by Proposition 4.18 we know that $N_{\mathfrak{G}}(\operatorname{Hom}(L, C_K))$ has finite index in $\operatorname{Hom}_{\mathfrak{G}}(L, C_K)$, so it follows that $T(\mathbb{A}_F)/T(F)$ has finite index in $\operatorname{Hom}_{\mathfrak{G}}(L, C_K)$. Therefore we have the following exact sequence

$$0 \longrightarrow T(\mathbb{A}_F)/T(F) \longrightarrow \operatorname{Hom}_{\mathfrak{G}}(L, C_K) \longrightarrow X \longrightarrow 0$$

where X is some finite group. If we then apply the functor $\operatorname{Hom}(-, D')$ (which is an exact functor since D' is $\mathbb{Z}$-injective) we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}(X, D') \longrightarrow \operatorname{Hom}(\operatorname{Hom}_{\mathfrak{G}}(L, C_K), D') \longrightarrow \operatorname{Hom}(T(\mathbb{A}_F)/T(F), D') \longrightarrow 0.$$

Now by assumption $\operatorname{Hom}(X, D')$ is a finite group, so it follows by Theorem 4.4 that we have a homomorphism from $H^1_{cts}(W_{K/F}, \widehat{T}_{D'})$ onto $\operatorname{Hom}_{cts}(T(\mathbb{A}_F)/T(F), D')$ with finite kernel.

Next we want prove that if D' is Hausdorff, then the kernel consists of locally trivial classes. With this in mind we use 4.24 and 4.3 to form the commutative diagram

$$H^1_{cts}(W_F, \widehat{T}_{D'}) \longrightarrow \operatorname{Hom}_{cts}(T(\mathbb{A}_F)/T(F), D')$$

$$\prod_v H^1_{cts}(W_{F_v}, \widehat{T}_{D'}) \xrightarrow{\ \sim\ } \prod_v \operatorname{Hom}_{cts}(T(F_v), D').$$

From part (a) we have that the lower horizontal arrow will be an isomorphism, and the result will follow if we can prove that the second vertical arrow is injective.

Let $\chi : T(\mathbb{A}_F)/T(F) \longrightarrow D'$ be a continuous homomorphism, whose restriction to $T(F_v)$ is the trivial homomorphism, for all places v. We want to show that this is in fact the trivial homomorphism. Note that we have

$$\bigoplus_v T(F_v) \subseteq \ker \chi.$$

Now since D' is Hausdorff, we have that $\ker \chi$ is closed in $T(\mathbb{A}_F)/T(F)$. Therefore the result will follow if we can show that $\bigoplus_v T(F_v)$ is dense in $T(\mathbb{A}_F)$, since this would mean that $T(\mathbb{A}_F)$ is also in the kernel of χ, which makes χ trivial.

Now, since $T(\mathbb{A}_F)$ is the restricted topological product of $T(F_v)$ with respect to $T(\mathcal{O}_v)$, it is easy to see that any non-empty open set in $T(\mathbb{A}_F)$ meets $\bigoplus_v T(F_v)$. Hence the result follows.

$\square$

A particular case of interest is when we set $D = \mathbb{C}_p^\times$, where by $\mathbb{C}_p$ we mean the completion of an algebraic closure of $\mathbb{Q}_p$. It is well known that as fields, $\mathbb{C}$ and $\mathbb{C}_p$ are isomorphic, but not as topological groups. As we have seen, the topology on D is irrelevant for Theorem 4.4, and is only needed for the last part of Theorem 4.25,

so since $\mathbb{C}_p^\times$ is a divisible abelian topological group and for any $n \in \mathbb{Z}_{>0}$ we have that the number of elements of order dividing n is finite, which means $\text{Hom}(X, \mathbb{C}_p^\times)$ will be a finite group for any finite group X, furthermore it is Hausdorff since it is a metric space. So we have that both Theorems apply, from which we can then deduce the abelian case of the p-adic Langlands program. In general we can let $D = A^\times$, where A is any Hausdorff topological field, like for example[9] $\overline{\mathbb{F}_p}, \overline{\mathbb{Q}_p}$ or $\mathbb{C}$. The case $D = \mathbb{C}^\times$ was what was originally proved by Langlands in [1].

4.5   Reformulation of Class Field Theory

To conclude, we will show very briefly, how Langlands' original paper relates to class field theory. It should be noted that the existence of the Weil group of a number field, encompasses many of the results of class field theory. Therefore what we have achieved is not an alternative proof of these results, but only a reformulation these results using representation theory.

We begin by looking at Theorem 4.4, and setting $L = \mathbb{Z}$ and $D = \mathbb{C}^\times$. With this we have that $\widehat{L} = \mathbb{Z}$ and $\widehat{T}_D = \mathbb{C}^\times$. Now let F be any field (local or global) and let K/F be a finite Galois extension. Then give $\widehat{L}$ and $\widehat{T}_D = \mathbb{C}^\times$, the trivial $\text{Gal}(K/F)$-action.

Now note that this means that $\text{Hom}(L, C_K) \cong \widehat{L} \otimes C_K \cong C_K$ and these are all $\text{Gal}(K/F)$-module isomorphisms, which means that $\text{Hom}_{\text{Gal}(K/F)}(L, C_K) \cong C_F$. Theorem 4.4 then tells us that we have an isomorphism

$$\Psi : H^1_{cts}(W_{K/F}, \mathbb{C}^\times) \longrightarrow \text{Hom}_{cts}(C_F, \mathbb{C}^\times).$$

The continuous homomorphisms $\chi \in \text{Hom}_{cts}(C_F, \mathbb{C}^\times)$ are called Hecke characters. Now since $W_{K/F}$ acts trivially on $\mathbb{C}^\times$, we have

$$H^1_{cts}(W_{K/F}, \mathbb{C}^\times) = \text{Hom}_{cts}(W_{K/F}, \mathbb{C}^\times).$$

Moreover, we have a surjective homomorphism

$$\text{Hom}_{cts}(W_{K/F}, \mathbb{C}^\times) \longrightarrow \text{Hom}_{cts}(\text{Gal}(K/F), \mathbb{C}^\times).$$

So if we combine these two homomorphism, what we get is a correspondence between the Hecke characters and 1-dimensional representations of $\text{Gal}(K/F)$. Now we can use this correspondence, and the fact that both C_F and $W_{K/F}$ are locally compact, to get a surjective homomorphism

$$C_F \longrightarrow \text{Gal}(K/F)^{ab},$$

as we did in class field theory (section 2).

Now we come to the question of how to use these ideas to generalize class field theory to non-abelian situations. The idea is to create an analogue of this correspondence between Hecke characters and 1-dimensional representations of $\text{Gal}(\overline{F}/F)$. Now to do this we need a replacement for the Hecke characters. To find this replacement, note that we can write $C_K = \mathbb{A}^\times/K^\times$ as $\text{GL}_1(\mathbb{A}_K)/\text{GL}_1(K)$, which is more commonly written as $\text{GL}_1(K)\backslash\text{GL}_1(\mathbb{A}_K)$. Now it is conjectured that one can find a correspondence between the n-dimensional representations of $\text{Gal}(\overline{F}/F)$, with what are called **automorphic representations** of

$$\text{GL}_n(K)\backslash\text{GL}_n(\mathbb{A}_K).$$

In order to properly define what an automorphic representation is, would take considerable work, so we will not do this here, since our intention is only to give a taste of how one uses Langlands' results to, in a way, generalize

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[9]Here $\overline{A}$, means the algebraic closure of A

class field theory. For further reading on this subject, we recommend the following introductory text [18].

It is using this idea of trying to relate n-dimensional representations of $\mathrm{Gal}(\overline{F}/F)$ with automorphic representations of $\mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)$, that forms the basis of the generalization of class field theory to non-abelian situations.

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