KMB, 2017-07-23

MSRI Summer School, Example Sheet.

1. (a) Check that the filtration $I_{L/K,i}$ on the inertia group is well-defined independent of a choice of uniformiser π_L.

(b) Check that $I_{L/K,i}/I_{L/K,i+1} \to P_L^i/P_L^{i+1}$ is a homomorphism (if $i=0$ interpret the right hand side as the units in the residue field of L).

2. Check that (arithmetic) Frobenius acts as multiplication by the size of k_K on $\mathrm{Gal}(K^t/K^{nr})$.

3. Outline the proof that if $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(\mathbf{Q}_\ell)$ is continuous and if $\ell \neq p$ then ρ must arise from a Weil-Deligne representation of $W_{\mathbf{Q}_p}$ (see Tate's article in Corvallis).

4. Show that any Weil-Deligne representation $(\rho_0, N) : W_{\mathbf{Q}_p} \to \mathrm{GL}_n(\mathbf{Q}_\ell)$ $(\ell \neq p)$ determines an ℓ-adic representation and that this is, up to isomorphism, independent of the choice of ϕ and t_L.

5. Show that if π is an admissible irreducible representation of $\mathrm{GL}_n(K)$ over $\mathbf{C}$ then the centre $K^\times$ of $\mathrm{GL}_n(K)$ acts by a character $\chi_\pi : K^\times \to \mathbf{C}^\times$.

6. Say $K/\mathbf{Q}_p$ is finite. Show that $B(K).\,\mathrm{GL}_2(\mathcal{O}_K) = \mathrm{GL}_2(K)$ (that is any element of $\mathrm{GL}_2(K)$ can be written as bk with b upper triangular and $k \in \mathrm{GL}_2(\mathcal{O}_K)$).

7. Let $\pi = I(\chi_1, \chi_2)$ with $\chi_1/\chi_2 \neq |\cdot|_K^{\pm 1}$. Assuming Q6, show that $\pi^{\mathrm{GL}_2(\mathcal{O}_K)}$ has dimension at most 1; show furthermore that it has dimension 1 precisely when χ_1 and χ_2 are unramified, and that in this case the Hecke operator T acts on $\pi^{\mathrm{GL}_2(\mathcal{O}_K)}$ via the scalar $q_K^{1/2}(\chi_1(\pi_K) + \chi_2(\pi_K))$.

8. Show that $\prod_p \mathrm{GL}_n(\mathbf{Z}_p) \times \mathrm{GL}_n(\mathbf{R})$ is an open subgroup of $\mathrm{GL}_n(\mathbf{A}_{\mathbf{Q}})$ with the usual toopology.

9. Let χ be an algebraic grossencharacter. Show that $E_\chi = \mathbf{Q}(\mathrm{Im}(\chi_0))$ is a number field.

10. If χ is an algebraic grossencharacter, show that χ_λ^G are a compatible system of λ-adic representations.

11. Is $\mathbf{Q}_p^{nr}$ complete?

12. If an ℓ-adic representation is pure, show that it is rational over $\overline{\mathbf{Q}}$.

13. Is a pure ℓ-adic representation necessarily rational over a number field?

14. Classify all Weil-Deligne representations $W_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{C})$, assuming $p > 2$. Find out what happens when $p = 2$ for extra credit.