Automorphic forms course, Oct–Dec 2005.

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Abstract

These are **rough notes** for a course I gave on automorphic forms Oct–Dec 2005.

Lecture 1 (19/10/5): I gave an overview which I didn't type up.
Lecture 2: (26/10/5) Now I'll start properly.

1 Group varieties.

Let k be a field. Working definition: a *group variety* over k is an algebraic variety G over k (that is, a scheme of finite type over $\mathrm{Spec}(k)$) equipped with multiplication, inversion, and identity $e \in G(k)$.

Example: the affine line, an elliptic curve, a torus. GL_n . Proof that GL_n is affine. Quick proof that $\mathbf{Z}$ is the endomorphisms of the torus, even in characteristic p.

A representation of a group G is a morphism of varieties $G \to \mathrm{GL}_n$. Example: the upper triangular representation of the affine line.

Let's assume also that G is smooth and that k is algebraically closed. Then there's a nice structure theorem, which goes as follows.

We have $G \supseteq G_0$, the connected component of the identity, and the quotient is finite. Then $G_0 \supseteq G_1$, the maximal connected affine subgroup; this is normal, and the quotient is projective (and hence an abelian variety) (this is a theorem of Chevalley, whose proof is not in the literature but is on the web). Now $G_1 \supseteq G_2$, the *radical* of G_1, which is the maximal connected affine solvable normal subgroup (solvable means the obvious thing because if G is connected and H, K are Zariski-closed subgroups, one of which is connected, then the commutator subgroup is also connected), and it also contains G_3, the maximal connected unipotent subgroup of G_2, which is normal in G_2 and in G_1. This is the *unipotent radical.* The quotient G_2/G_3 is a torus. Unipotent means all representations have $\rho(g) - 1$ nilpotent.

Néron models are extensions of abelian varieties by tori.

Say G is *reductive* if G is affine and $G_3 = 0$, so it's an affine algebraic group with no unipotent radical. I've never understood Richard Thomas' definition of reductive.

An algebraic variety has a tangent space at a point, which is a finite-dimensional vector space over the residue field of the point. The tangent space of a group variety at the origin inherits the structure of an abstract Lie algebra (and in characteristic p it gets a little more). The tangent space of a semisimple group variety is a semisimple Lie algebra, and there's a classification of semisimple Lie algebras which is usually presented over the complexes but works over any field. The classification is in terms of root systems.

But you have to be careful: SL_n and PGL_n have the same tangent space over $\mathbf{C}$ (there is a finite unramified covering $\mathrm{SL}_n \to \mathrm{PGL}_n$ of degree n), and are both semisimple groups, but they aren't isomorphic; (consider centres). The tangent space doesn't see nice finite covers, so you can't reconstruct a group variety from its tangent space, even if the group variety is connected. On the other hand a mild generalisation of root systems, called root data, can see these "finite errors" and can also deal with the torus kicking around.

A *root datum* is a 4-tuple $(X, \Psi, X^\vee, \Psi^\vee)$ such that X and $X^\vee$ are finitely-generated free abelian groups, equipped with a perfect pairing $\langle , \rangle : X \times X^\vee \to \mathbf{Z}$, where $\Psi \subseteq X$ and $\Psi^\vee \subseteq X^\vee$ are finite subsets equipped with a bijection $\Psi \to \Psi^\vee$ denoted $\alpha \mapsto \alpha^\vee$, and two axioms.

The first is $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Psi$. This implies that for $\alpha \in \Psi$, the map $s_\alpha : X \to X$ defined by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ satisfies $s_\alpha^2 = 1$, and similarly $s_{\alpha^\vee}(u) = u - \langle \alpha, u \rangle \alpha^\vee$. The second axiom is that for all α we have $s_\alpha(\Psi) = \Psi$ and $s_\alpha(\Psi^\vee) = \Psi^\vee$.

Examples: $X = \mathbf{Z} = X^\vee$ with $\langle , \rangle$ the product. Then we can have $\Psi = \{\pm 1\}$, or $\{\pm 2\}$, or $\{\pm 1, \pm 2\}$, or empty. Empty is a possibility, so these aren't root systems, they are something a bit more general. That one with 1 and 2 in it is lousy though: say a root datum is *reduced* if $\alpha \in \Psi$ and $\lambda\alpha \in \Psi$ for $\lambda \in \mathbf{Q}$ implies $\lambda = \pm 1$.

Let Q be the $\mathbf{Z}$-span of Ψ and let V be the $\mathbf{Q}$-span. Similarly let X_0 be $(\Psi^\vee)^\perp$ and let V_0 be the $\mathbf{Q}$-span of X_0. Then $Q + X_0 = Q \oplus X_0 \subseteq X$ has finite index. Say that the root datum is *semisimple* if $X_0 = 0$ and *toric* if $\Psi = \emptyset$. Note that Ψ is a root system in V. One can classify root systems completely: there's irreducible ones A_n $(n \geq 1)$ and B_n $(n \geq 2)$ and C_n $(n \geq 3)$ and D_n $(n \geq 4)$ and E_6, E_7, E_8, F_4, G_2 (and a non-reduced one BC_n, $n \geq 1$) and every root system is a direct sum of finitely many irreducible ones. In our situation we have an injection $X \to S \oplus T$ with S semisimple and T toric, and the cokernel is finite. So we have a classification of root datums, which is useful enough in practice.

Here's the big construction: let G be connected and reductive, and let k be algebraically closed. Choose a maximal torus T (probably not normal). Set $X = \mathrm{Hom}(T, \mathbf{G}_m)$ and $X^\vee = \mathrm{Hom}(\mathbf{G}_m, T)$. Let Ψ be the root system for (G, T) (the characters that show up in the action of T on the tangent space). The definition of $\Psi^\vee$ is messier. If $\alpha \in \Psi$ then let T_α be the identity component of $\ker(\alpha)$; this is contained in T and is a little smaller. Let Z_α denote the centraliser of T_α in G; this is now a little bigger than T. Let G_α denote its derived subgroup; one can check that this is either SL_2 or PGL_2 because one can characterise semisimple groups of rank 1. In either case, an explicit calculation shows that there's a unique map $\alpha^\vee : \mathbf{G}_m \to G_\alpha$ such that T is generated by T_α and the image of $\alpha^\vee$ and furthermore that $\langle \alpha, \alpha^\vee \rangle = 2$. So there we go. This gives us a reduced root datum associated to any connected reductive algebraic group (plus torus) and the big theorem is that this establishes a bijection between reduced root datums and connected reductive algebraic groups. The hard part is of course the construction of the group from the data. You reduce to the cases semisimple and toric. The torus is easy. The semisimple case is again a theorem of Chevalley from 1960 or so.

Hence one can write down "all" reductive groups over an algebraically closed field.

Lecture 3: (2/11/5)

Refs for previous lecture were: "semi-simple algebraic groups" by Kneser in Cassels-Froehlich, Borel's "linear algebraic groups" for the basics, and Springer's "Reductive groups" in Corvallis.

k alg closed still. Fact: any affine group is a closed subgroup of some GL_n. A representation of G is a map $G \to \mathrm{GL}_n$. A group is unipotent if all its reps are unipotent. If G is affine then it has a unique maximal connected solvable (resp. unipotent) normal subgroup, the (resp. unipotent) radical. G is called reductive if it's affine and has no unipotent radical, and semisimple if it has no radical. Recall: connected reductive groups are classified by a piece of linear algebra data, called the root datum associated to G. This wasn't an equivalence of categories, just a bijection on objects: a group has lots more automorphisms than the piece of linear algebra data, for example.

An isogeny $G \to H$ is a surjection with finite kernel. An isogeny of root data is an injective group hom $X' \to X$ with finite cokernel inducing a bijection $\Phi' \to \Phi$ and such that the dual of the hom induces a bijection $\Phi^\vee \to \Phi'^\vee$. It is the case that isogenies of root data give rise to isogenies between groups.

Using this classification one can prove that up to isogeny, any connected reductive G is a product of a semisimple group and a torus. Tori are easy, they are products of $\mathbf{G}_m$. Semisimple groups are again not too hard, they are, up to finite groups, products of simple groups, which, using the root datum side of the picture, one can completely write down. Every simple group is isogenous to one of A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2, and the resulting groups are

$A_n = \mathrm{SL}_{n+1}$, B_n and D_n are orthogonal groups, C_n is Sp_{2n}, and the other five are exceptional groups.

If k is perfect, but not algebraically closed, then you define reductive and torus by "becomes reductive/toric over the alg closure". Note that here there's an issue with the classification because there are groups over k which are not isomorphic, but which become isomorphic over an algebraic closure. For example the closed subgroup of 2-space over the reals defined by $x^2 + y^2 = 1$ with multiplication $(a,b)(c,d) = (ac - bd, ad + bc)$ [think $(a,b) = a + bi$; then $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$] is certainly not isomorphic to $\mathbf{G}_m$, but it becomes isomorphic to $\mathbf{G}_m$ over the complexes. Similarly, if K is a field and L is either a separable quadratic field extension of K or $K \oplus K$, whose Galois group has non-trivial element $*$ (which is the switch $(a,b)^* = (b,a)$ in the second case) then one can define the unitary group $U(n)/K$ as the subgroup of $\mathrm{GL}_n(L)$ consisting of matrices M such that $M.M^{*t} = 1$. This is a bunch of equations which aren't L-linear but which are K-linear so the resulting algebraic variety is defined over K but not L. This group is typically not isomorphic to GL_n. On the other hand if $L = K \oplus K$ then it is; however this is always the case over an algebraically closed field so $U(n)$ is a form of GL_n.

For tori, one can work out what's going on explicitly. Given a torus over k, G_k (the absolute Galois group of k) acts (continuously) on $\mathrm{Hom}_{\overline{k}}(T, \mathbf{G}_m) = \mathbf{Z}^n$ and so we get an induced map of G_k on a finite free abelian group. The converse is also true: given the action we recover the torus, and this is an anti-equivalence of categories.

More generally, given a group G/k, the computation of all forms of G is an exercise in Galois cohomology; $\mathrm{Gal}(\overline{k}/k)$ acts on $G(\overline{k})$ and there's a cohomology pointed set $H^1(G_k, \mathrm{Aut}(G))$ (continuous cocycles modulo equivalence). One could try and work such a thing out in any given case. The theory of automorphic forms for G will change massively if you replace G by a form of G. One of Langlands' amazing observations is that the theories of automorphic forms for distinct forms of the same group should be strongly related.

2 Automorphic forms on real reductive groups.

I promised a definition so here is one.

I spent a lot of time trying to learn some analysis but realised later on that most of it was irrelevant. Ref is Borel-Jacquet in Corvallis. Let G be a reductive group over $\mathbf{Q}$ and K a maximal compact subgroup of $G(\mathbf{R})$. We want to consider C^∞ functions on $G(\mathbf{R})$ but we want to only consider the ones of interest to us, so we need boundedness conditions. Turns out we need other conditions too which, if I were an expert, I'd tell you why we needed them.

Let $\mathbf{M}_n$ be the group of $n \times n$ matrices under addition, and consider GL_n as an open subvariety of M_n (of course it's not a subgroup). Choose $\sigma : G \to \mathrm{GL}_n$ an injection *with closed image in* $\mathbf{M}_n$. Note that this is strictly stronger than just demanding the image is closed (for example if $G = \mathrm{GL}_1$ then the identity map $G \to \mathrm{GL}_1$ has image closed in GL_1 but not in M_1). Note also however that if $\sigma : G \to \mathrm{GL}_n$ has image closed in GL_n then adding either $\det(\sigma)^{-1}$ as one new coordinate, or $\sigma(g^{-1})^t$ as n new coordinates, gives a new representation of degree $n + 1$ or $2n$ which does have the required property.

Anyway, with σ as above, if $g \in G(\mathbf{R})$ then define $||g||_\sigma$ to be $\mathrm{tr}(\sigma(g)^t\sigma(g))^{1/2}$. This depends on σ but in fact I am told that[1] for any σ and τ there exists C and n such that $||x||_\sigma \leq C||x||_\tau^n$. We say $f : G(\mathbf{R}) \to \mathbf{C}$ is *slowly increasing* if there's a norm on $G(\mathbf{R})$ and C and n such that $|f(x)| \leq C||x||^n$ for all x.

Now let Γ be an "arithmetic subgroup" of $G(\mathbf{Q})$; this I think means the intersection of $G(\mathbf{Q})$ with some compact open in $G(\mathbf{A}_f)$. An automorphic form for (G, K, Γ) is a smooth complex-valued function $f : G(\mathbf{R}) \to \mathbf{C}$ such that

(a) $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$,

(b) f is right K-finite, (this means that the translates of f of the form $x \mapsto f(xk)$ generate a finite-dimensional vector space),

(c) There's an ideal of finite codimension in $Z(U(\mathfrak{g}_\mathbf{C}))$ which annihiliates f (I'll define this in the next section)

(d) f is slowly increasing.

[1] I should do this exercise.

I need to explain what (c) means.
Lecture 4: (9/11/5)

3 Lie groups, Lie algebras, Universal enveloping algebras.

Let G be a group variety over the reals; then $G(\mathbf{R})$ is a real Lie group (which is just a group in the category of real manifolds). I always have to get my head around manifolds, so let me do this now. Given a point p on a manifold M we have the space of germs of smooth functions at this point p, called $C_p(M)$, and the tangent space to M at p is just the linear maps $L_p : C_p(M) \to \mathbf{R}$ such that $L_p(fg) = f(p)L_p(g) + g(p)L_p(f)$. A local calculation shows that this space is finite-dimensional. Tangent spaces glue together to give you the tangent bundle. A smooth vector field is just a smooth section of this tangent bundle. This is a big real vector space; an element of it is, for each point p, a linear map $L_p : C_p(M) \to \mathbf{R}$ such that if f is in $C^\infty(M)$ then the function sending p to $L_p(f)$ is also in $C^\infty(M)$. Note that L is determined by its action on $C^\infty(M)$, by some partition of unity argument, and that $L(fg) = fL(g) + gL(f)$. I guess this gives a bijection between smooth vector fields and derivations of $C^\infty(M)$.

A weird calculation shows that if L and M are smooth vector fields then LM isn't, but $LM - ML$ is. Let's call this $[L, M]$. This puts the structure of a Lie algebra on the space of all smooth vector fields.

Formal definition: k any field. A Lie algebra over k is a vector space V/k equipped with an alternating bilinear map $[,] : V \times V \to V$ such that $[v, v] = 0$ for all v and x $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$. Natural example: if A is an associative algebra over k then define $[a, b] = ab - ba$. Note that not all Lie algebras arise in this way, and many arise more than once.

If G is a Lie group then one can pull back vector fields by the action of the group and ask which are left invariant. A left invariant vector field is determined by L_0 which establishes a bijection between left invariant vector fields and $T_0(M)$. So magically $T_0(M)$ inherits the structure of a finite-dimensional Lie algebra. In practice if $G = \mathrm{GL}_n(\mathbf{R})$ then $T_0(G) = M_n(\mathbf{R})$ and the Lie algebra structure turns out to be $AB - BA$ by a local calculation. Now if G is a closed subgroup of $\mathrm{GL}_n(\mathbf{R})$ then the tangent space is "what you think it is" and the Lie algebra is defined by $XY - YX$ again, but XY might not be in the Lie algebra. For example SL_n.

All this can be done algebraically; if G is an affine group variety then its algebraic tangent space gets the structure of a Lie algebra over the base field. That's why the classifications are related.

There is a notion of solvable and nilpotent Lie algebras, of "ideals" (which correspond to normal subgroups) and then radicals (largest solvable ideal) and nilpotent radicals, and semisimple, simple, and reductive Lie algebras. The analogue of a torus is an abelian Lie algebra, which has $[v, w] = 0$ for all v and w. The analogue of a subtorus is a Cartan subalgebra, which is nilpotent subalgebra equal to its own normaliser. A reductive Lie algebra is the direct sum of a semisimple one and an abelian one. There's a classification of Lie algebras over $\mathbf{C}$ and there's a simple Lie algebras in terms of root data just as before. A reductive affine algebraic group over an alg closed field is determined up to isogeny by its Lie algebra because one can reconstruct the root datum tensored with $\mathbf{Q}$.

Now the universal enveloping algebra. The construction $[a, b] = ab - ba$ gives a morphism of functors from associative algebras to Lie algebras over k. This functor has a left adjoint, by which I mean the following: there is an associative algebra $U(\mathfrak{g})$ that one can attach to a Lie algebra $\mathfrak{g}$, with the property that $\mathrm{Hom}(\mathfrak{g}, A) = \mathrm{Hom}(U(\mathfrak{g}), A)$ where the first is Lie algebra homs and the second associative algebra homs. The existence of $U(\mathfrak{g})$ follows from the general adjoint functor theorem, but, more usefully, one can just build $U(\mathfrak{g})$ as $k \oplus \mathfrak{g} \oplus T^2(\mathfrak{g}) \oplus \cdots$ modulo the bi-ideal generated by $v \otimes w - w \otimes v - [v, w]$, where $T^n(\mathfrak{g})$ is the tensor algebra $g \otimes_k \mathfrak{g} \otimes_k \cdots$. This explicit construction can be used to show that Poincaré–Birkhoff–Witt theorem, which says that if $\mathfrak{g}$ is finite-dimensional and $e_1, \ldots, e_n$ is a basis of $\mathfrak{g}$ then $e_1^{a_1} e_2^{a_2} \ldots$ are a basis for $U(\mathfrak{g})$. Example: $\mathfrak{g}$ abelian (that is, $[v, w] = 0$ for all v and w); then $U(\mathfrak{g})$ is just non-canonically a polynomial algebra. If however $\mathfrak{g}$ is something interesting like $\mathfrak{sl}_2$ then we have a basis $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$

and $h = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and then $[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$. The universal enveloping algebra is some non-commutative ring generated by E, F and H, where you forget that these are matrices: these elements satisfy satisfying $EF - FE = H$, but $E^2 \neq 0$ (by PBW!). The adjoint representation of $\mathfrak{g}$ (with ordered basis e, f, h) defined by $g \mapsto (h \mapsto [g,h])$ is

$$e \mapsto \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$f \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}$$

$$h \mapsto \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the Killing Form, $B(x,y) = \mathrm{tr}(\rho(x)\rho(y))$, with ρ the adjoint representation, is represented by the matrix

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Hence $e^* = f/4$, $f^* = e/4$ and $h^* = h/8$. So $EE^* + FF^* + HH^* = (2EF + 2FE + H^2)/8$ will be in the centre of the universal enveloping algebra. Set $\Omega = 2EF + 2FE + H^2 + 1 = 4EF + H^2 - 2H + 1$. Fun exercise: Ω is in the centre. Much less obvious: $Z(U(\mathfrak{sl}_2))$ is $\mathbf{C}[\Omega]$. In fact if $\mathfrak{g}/\mathbf{C}$ is reductive then $Z(U(\mathfrak{g}))$ is a polynomial algebra over $\mathbf{C}$ in ℓ variables, where ℓ is the rank of the Lie algebra (the dimension of a Cartan subalgebra). See Dixmier's book, Theorem 7.3.8(ii) for the semisimple case; the extension to the reductive case is elementary (unless I made a mistake!).

Note that $\mathfrak{g}$ acts on $C^\infty(M)$ and the endomorphisms form a ring, so $U(\mathfrak{g})$ acts too. One can think of elements of $\mathfrak{g}$ as first order differential operators.

Now our definition of automorphic form makes sense.

Lecture 5: (16/11/05)

4 A basic theorem about automorphic forms.

If we fix an irreducible finite-dimensional representation ρ of K then we can say that f (an automorphic form) is of type ρ if the induced representation of K is isomorphic to a finite sum of copies of ρ. If G is connected reductive over $\mathbf{Q}$ then $A(\Gamma, \rho, J, K)$ is the automorphic forms as above (J an ideal of finite index in the centre of the universal env alg).

Harish-Chandra proved this space was finite-dimensional! I don't know the proof.

Another point of view: if $r : K \to \mathrm{GL}(V)$ is an irreducible finite-dimensional complex unitary representation of K (on an inner product space V) then a V-valued automorphic form is $\phi : G(\mathbf{R}) \to V$ satisfying $\phi(\gamma x) = \phi(x)$ for all $\gamma \in \Gamma$, existence of J of finite codimension, slowly increasing, and now $\phi(xk) = r(k^{-1})\phi(x)$. Let $A(\Gamma, V, J, K)$ denote this space. For any $v \in V$ the function $f : G(\mathbf{R}) \to \mathbf{C}$ defined by $f(x) = (\phi(x), v)$ is a scalar automorphic form of type r, because $f(xk) = (\phi(xk), v) = (k^{-1}\phi(x), v) = (\phi(x), kv)$, and so the space of right K-translates of f admites a K-invariant surjection from V and hence this space is finite-dimensional. Conversely given an automorphic form f of type r, consider the space spanned by the right translates of K on f. This is finite-dimensional, by assumption, and isomorphic to a direct sum of copies of r. Choose $k_1, \ldots, k_n$ such that $g \mapsto f(gk_i)$ is a basis of this space. Now define $\phi : G(\mathbf{R}) \to \mathbf{C}^n$ by $\phi(g) = (\overline{f(gk_1)}, \overline{f(gk_2)}, \ldots)$. This space affords a representation of K, which we know is isomorphic to a direct sum of copies of V. Break the space up into irreducibles $\mathbf{C}^n = V \oplus V \oplus \ldots \oplus V$, and let π_i denote the projection onto the ith factor. Put an inner product on $\mathbf{C}^n$ to make everything unitary. We recover f as the composite $G(\mathbf{R}) \to \mathbf{C}^n \to \mathbf{C}$, this last map being linear, so it breaks up as a sum over i of $G(\mathbf{R}) \to V \to \mathbf{C}$. Explicitly, there are V-valued automorphic forms f_i and vectors v_i such that $f(g) = \sum_i (f_i(g), v_i)$. We have proved that the natural map $\overline{V} \times_{\mathbf{C}} A(\Gamma, V, J, K) \to A(\Gamma, r, J, K)$ sending (v, ϕ) to $g \mapsto (v, \phi(g))_V$ is a surjection, and I got tangled up trying to prove it was an isomorphism. Is it?? Probably.

5 Explicit examples.

Let's compute explicitly what's going on for GL_1. $K = \{\pm 1\}$ and $\mathfrak{g}$ is 1-dimensional so it must be $D := Xd/dX$ so $U(\mathfrak{g}_{\mathbf{C}})$ is $C[D]$. Any ideal of finite index J is principal. If we fix a representation of K then this tells us $f(-x)$ in terms of $f(x)$ so we may as well consider only $x > 0$; then we can take logs and get functions on $\mathbf{R}$. Because everything is an isomorphism of Lie groups we deduce that D becomes (some constant times) d/dt. 6th form stuff (and perhaps something non-trivial: I need to know that any C^∞ function which is killed by an ODE with constant coeffts is analytic?) now tells us that the space of solutions is finite-dimensional and has a basis consisting of $t^n \exp(st)$ for s a complex number and $n \geq 0$ an integer. So back in the x world it's $\log(x)^n x^s$ for $s \in \mathbf{C}$. Now which of these are slowly increasing? $||x|| = |x + 1/x|$ because $\sigma(x) = \mathrm{diag}(x, x^{-1}, 1, 1)$ will do and so we're home. We see the space is finite-dimensional and furthermore we become slightly bewildered by the general convention that we should fix a central character.

Lecture 6: (23/11/05)

The most general thing that people seem to do is to consider Z, the maximal split torus of the centre of G over $\mathbf{Q}$, and demand that $f(zx) = \chi(z)f(x)$ for some character, or quasi-character, χ of $Z(\mathbf{R})$ (not necessarily algebraic). This just seems to me to throw away the logs.

Another example: the norm 1 units in $\mathbf{Q}(i)$. That is, $x^2 + y^2 = 1$, the non-split torus. Then $G(\mathbf{R}) = S^1 = K$ and $\Gamma = 1$ so the situation is quite different here: if we choose a representation of K, namely $(x, y) \mapsto z^n$ with $z = x + iy$ and $n \in \mathbf{Z}$ then it basically uniquely determines f; we have $f(z) = cz^n$ for $c \in \mathbf{C}$ and that's it, really, isn't it. The Lie algebra is the reals again, and if we think of the unit circle as $e^{i\theta}$ with θ real, the invariant diff op is $d/d\theta$ and $f(\theta) = ce^{in\theta}$ so $Df = inf$ and hence most choices of J give a zero-dimensional space, whatever the representation of K. Boundnedness: we can take our representation to send (x, y) to $(x, y; -y, x)$ and the norm is constant. So slowly increasing is the same as bounded and indeed our functions are bounded.

Next, $G = \mathrm{GL}_2$. Consider the j-function. Define $j(-\tau) = j(\tau)$ and this extends j to a function on $\mathbf{C} \backslash \mathbf{R}$. Now set $f(a, b; c, d) = j((ai + b)/(ci + d))$, that is, $f(\gamma) = j(\gamma i)$. Is this an automorphic form? It satisfies (a) and (b). The universal enveloping algebra is $\mathbf{C}[\Omega, Z]$ and we have to understand how Ω acts. To do this we use a trick. The Lie algebra is usually acting as left-invariant differential operators and, as far as I can see, they don't even induce differential operators on functions on the upper half plane. For example $E = (0, 1; 0, 0)$ acts thus. If $f : \mathrm{GL}_2(\mathbf{R}) \to \mathbf{C}$ and X is in the Lie algebra then $(Xf)(g) = (d/dt)f(ge^{Xt})$ so if $f(kg) = f(g)$ then $(Xf)(kg) = Xf(g)$. But we can't just switch our conventions, unfortunately: we've let $U(\mathfrak{g})$ act on the left and G act on the left. Fortunately, we are saved by the basic fact that the centre of $U(\mathfrak{g})$ can be identified with the bi-invariant differential operators! Recall that $g \in \mathfrak{g}$ gives a map $C^\infty(G) \to C^\infty(G)$ which commutes with $L_x : f \mapsto (g \mapsto f(xg))$ for all x. The basic fact, whose proof I don't know, is that the stuff in the centre is exactly the maps which commute with R_x too. So now we let the Lie algebra act via $(X^*f))(g) = (d/dt)f(e^{-Xt}g)$. These derivations generate an algebra anti-isomorphic to $U(\mathfrak{g})$ and we work in here instead. The point is that these operators induce differential operators on the upper half plane.

$E^*f = -df/dx$ (partial). F^*f is the derivative of $f(\tau/(-t\tau + 1)) = f(\tau + t\tau^2 + \ldots) = f(\tau + t(x^2 - y^2) + it(2xy) + \ldots)$ so $F^*f = (x^2 - y^2)f_x + 2xyf_y$. Exercise: $H^*f = -2xf_x - 2yf_y$ and $Z^*f = 0$ where $H = (1, 0; 0, -1)$ and Z is the identity. So $H^2 + 2FE + 2EF$ works out to be $2y^2(f_{xx} + f_{yy})$. In particular the function coming from the j-function is indeed an eigenvector for Ω. It's also an eigenvector for Z, the last remaining generator of the centre.

Stupid thing though: f is not slowly increasing. Indeed if we consider the matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ then it has norm λ for $\lambda >> 0$, but $f(\lambda)$ is $j(i\lambda)$ which is approximately $q^{-1} + \cdots = e^{2\pi\lambda}$ which is growing faster than any polynomial in λ. So (apart from the constant functions, which obviously work) we don't have an example of an automorphic form for GL_2 yet.

Lecture 7: (30/11/05)

I didn't really prepare this lecture.

Definition: a Maass form for $\mathrm{SL}_2(\mathbf{Z})$, is a C^∞ function on the upper half plane such that $f(\gamma z) = f(z)$ for $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, such that $f(x+iy) = O(y^N)$, and (crucially) f is an eigenfunction for $f \mapsto y^2(f_{xx} + f_{yy})$.

Given such an f we can build an automorphic form for GL_2 as above, and this time our axioms do work because we've put the boundedness criterion in. Note that if f were holomorphic then it would induce a holomorphic function on the quotient $Y_0(1)$ of the upper half plane by $\mathrm{SL}_2(\mathbf{Z})$, and the boundedness criterion shows that this function extends to $X_0(1)$, so it's constant.

So we've still not seen any examples. But here's one: fix $s \in \mathbf{C}$ with $\Re(s) > 1$ and define $E_s(\tau) = \sum'_{m,n\in\mathbf{Z}} \frac{y^s}{|m\tau+n|^{2s}}$, where the prime means m, n not both zero, and $\tau = x + iy$. This converges because $\Re(s) > 1$ and it's not too difficult to check it's bounded and $\mathrm{SL}_2(\mathbf{Z})$-invariant. The neat thing is that our differential operator is $\mathrm{GL}_2^+(\mathbf{R})$-invariant and y^s is an eigenvector for it, with eigenvalue $s(s-1)$, so $\tau \mapsto (\Im(\gamma\tau))^s$ is too, for any $\gamma \in \mathrm{GL}_2^+(\mathbf{R})$, and this is $y^s/|c\tau+d|^{2s}$. So E_s, being a sum of things of this form, is also an eigenvector for $y^2(d/dx^2 + d^2/dy^2)$, with eigenvalue $s(s-1)$.

Remark: it's not cuspidal though. And you can't multiply two together to get another one, so we can't generate all Maass forms this way. Indeed I don't think it's known how to generate all Maass forms. Gelbart writes down an eigenvalue for which there is a Maass form which isn't an Eisenstein series; he builds the form (following Maass) via a Grossencharacter of a real quadratic field.

6 The real definition of an automorphic form.

First let me mention restriction of scalars. If L/K is a finite extension of fields, and X is an affine variety over L, then $Y := \mathrm{Res}_{L/K}(X)$ is a variety over K with the property that $Y(M) = X(M \otimes_K L)$ for any K-algebra M. In particular $Y(M) = X(L)$. If X is a reductive group over L then Y is a reductive group over K.

Adeles: The adeles $\mathbf{A}_F$ of a number field F are the restricted product of the completions of F at all places. The finite adeles $\mathbf{A}_F^f$ are the restricted product over all finite places. We drop the F if it's $\mathbf{Q}$. Example: $\mathbf{A}^f = \widehat{\mathbf{Z}} \otimes \mathbf{Q}$ and $\mathbf{A} = \mathbf{A}^f \times \mathbf{R}$ and $\mathbf{A}_F = \mathbf{A} \otimes_{\mathbf{Q}} F$.

Now let G be a reductive group over $\mathbf{Q}$. Fix a maximal compact K_∞ of $G(\mathbf{R})$. Then $G(\mathbf{A}) = G(\mathbf{A}^f) \times G(\mathbf{R})$. A $\mathbf{C}$-valued function on $G(\mathbf{A})$ is *smooth* if it's locally constant at the finite places and C^∞ at the infinite places. In other words, if $x \in G(\mathbf{A}^f)$ and $y \in G(\mathbf{R})$ and we think of f as a function of two variables then, for fixed x we have $y \mapsto f(x,y)$ is C^∞, and for fixed y we have $x \mapsto f(x,y)$ is locally constant.

The real definition:

An automorphic form f for G and K_∞ is a smooth function f on $G(\mathbf{A})$ such that

(a) $f(\gamma x) = f(x)$ for all $\gamma \in G(\mathbf{Q})$

(b1) There's a max compact K in $G(\mathbf{A}^f)$ such that $f(xk) = f(x)$ for all $k \in K_\infty$

(b2) The translates of f under K_∞ form a finite-dimensional space

(c) There's an ideal J of finite codimension of $Z(U(\mathfrak{g}_{\infty,\mathbf{C}}))$ annihiliating f

(d) For each $x \in G(\mathbf{A}_f)$ the function $y \mapsto f(x,y)$ is slowly increasing as a function on $G(\mathbf{R})$.

We can refine (b1) a bit: say ρ is an irred rep of K_∞; then we could furthermore demand that the translates of f generate a finite-dimensional space which is iso to a direct sum of copies of ρ. Let $A(K, \rho, J, K_\infty)$ denote this space.

Claim: this space is finite-dimensional.

One can deduce this from the earlier claim about finite-dimensionality. Here's how. One needs some kind of finiteness theorem about the groups $G(\mathbf{A}^f)$; we're putting conditions on the finite parts of the functions by demanding left invariance under $G(\mathbf{Q})$ and right invariance under K. The theorem is

Theorem (Borel: Theorem 5.1 of "Some finiteness properties of adele groups over number fields" Publ math IHES 1963) $G(\mathbf{A}^f)$ is a *finite* disjoint union of double cosets $G(\mathbf{Q})c_iK$.

Remark: the proof works for an arbitrary "algebraic matric group" which I think is a closed subgroup of GL_n, so it works for arbitrary affine algebraic groups over $\mathbf{Q}$. Remark: Borel considers an embedding and only lets K be the stabiliser of a lattice, but it's OK.

Example: $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$. Then we need to check finiteness of

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A}_F)/\mathrm{GL}_n(F_\infty)\mathrm{GL}_n(\widehat{\mathcal{O}}).$$

This is just $\mathrm{GL}_n(F)\backslash$ $\widehat{\mathcal{O}}$-lattices in $(\mathbf{A}_F^f)^n$ which is $\mathrm{GL}_n(F)\backslash$ projective $\mathcal{O}_F$-modules in F^n which is iso classes of rank n projective $\mathcal{O}_F$-modules which is the class group of F.

Corollary: $A(K, \rho, J, K_\infty)$ is finite-dimensional

Proof: let C be the set of all the c_i. For $c \in C$ set $\Gamma_c := G(\mathbf{Q}) \cap cKc^{-1}$ in $G(\mathbf{A}^f)$: an arithmetic subgroup of $G(\mathbf{Q})$. Now here's the dictionary. By elementary group theory one checks easily that there's a bijection between $G(F)\backslash G(\mathbf{A})/K$ and the disjoint union over $c \in C$ of $\Gamma_c\backslash G(\mathbf{R})$, the dictionary being sending y in the cth term on the right to cy on the left. This map then induces an isomorphism

$$A(K, \rho, J, K_\infty) \cong \oplus_{c\in C} A(\Gamma_c, \rho, J, K_\infty).$$

sending f to the functions f_c on $G(\mathbf{R})$ defined by $f_c(x) = f(cx)$.

So really we're just using a fancier language to describe things we've seen before. But in fact the adelic language is more natural. I've not talked about Hecke operators but they are a pain to do on the right hand side—if $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_1$ then the old way of thinking about things via $G(\mathbf{R})$ gives us trouble because there aren't naturally elements of $G(\mathbf{Q})$ corresponding to an arbitrary prime ideal. But these things do exist in $(\mathbf{A})$.

Let's revisit GL_1 over $\mathbf{Q}$. Now K can be the things in $\widehat{\mathbf{Z}}^\times$ congruent to 1 mod N, and C is just $(\mathbf{Z}/N\mathbf{Z})/\pm 1$. If we fix J to be $(td/dt - s)$ then this forces the components at infinity to look like $x \mapsto x^s$ but instead of the 2-dimensional space we had before, we've got a $\phi(N)$-dimensional space and this space has a basis corresponding to the characters of $(\mathbf{Z}/N\mathbf{Z})^\times$. In general, for GL_1 over any number field, one can check that Grossencharacters, that is, continuous group homomorphisms $F^\times\backslash\mathbf{A}_F^\times \to \mathbf{C}^\times$, are examples of automorphic forms.

Now let's revisit GL_2. [see HMF notes]

Final lecture: 14/12/05.

One can characterise the space. It's not all the auto forms $A(K, \rho, J, K_\infty)$ for J =blah and ρ =2-dimensional because there's this extra condition that $phi(xr(\theta)) = e^{-ik\theta}$ so it's a very specific subspace. Note that this is not completely formal. To go the other way you have to do some analysis; you know that the function on the upper half plane that you're trying to construct satisfies a certain differential equation, and you have some other facts, and you need to deduce that it's holomorphic.

We've now seen a definition of an automorphic form. There are also vector-valued ones: if (ρ, V) is a finite-dimensional representation of K_∞ we can consider $f : G(\mathbf{A}) \to V$ such that $f(gu) = u^{-1}f(g)$ and these two definitions boil down to the same thing although the precise relation still eludes me.

An automorphic form is left-invariant under $G(\mathbf{Q})$ and right-invariant under K and transforms well under K_∞. You can also add a condition about how the centre acts (or how the maximal $\mathbf{Q}$-split torus in the centre acts).