# "Diophantine problems and $p$-adic period mappings" by B. Lawrence and A. Venkatesh

Study group organised by Ana Caraiani and Alexei Skorobogatov

The paper [LV] uses $p$-adic analytic methods (more precisely the period map defined by a family of varieties) to prove two kinds of results. Let $K$ be a number field, let $S$ be a finite set of places of $K$ containing the archimedean places, and let $\mathcal{O}_S$ be the ring of $S$-integers of $K$. Let $Y$ be a smooth, separated $\mathcal{O}_S$-scheme of finite type with generic fibre $Y_K$.

(a) When $Y_K$ is a hyperbolic curve, they prove the finiteness of $Y(\mathcal{O}_S)$. This gives a new proof of Faltings' theorem (Mordell conjecture) and of Siegel's theorem (the $S$-unit equation).

(b) When $Y_K$ is higher-dimensional, they give sufficient conditions for the finiteness of $Y(\mathcal{O}_S)$ and show that these conditions are satisfied for the scheme of moduli of smooth hypersurfaces of degree $d$ in $\mathbb{P}^n$ if $n$ is large and $d$ is larger than some bound depending on $n$. This relies on an Ax–Schanuel theorem recently proved by Bakker and Tsimerman.

The idea is to consider a proper smooth morphism $\pi : X \to Y$ of $\mathcal{O}_S$-schemes. Let $p$ be a prime which is coprime to the primes of $S$. For $y \in Y(K)$ denote by $X_y$ the fibre of $\pi$ at $y$. Consider the representation

$$\rho_y : G_K = \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}\, \mathrm{H}^i_{\text{ét}}((X_y)_{\overline{K}}, \mathbb{Q}_p).$$

An easy lemma of Faltings says that the semisimplification $\rho_y^{\mathrm{ss}}$ is isomorphic to one of finitely many representations. Let $v$ be a place of $K$ above $p$, let $G_{K_v} = \mathrm{Gal}(\overline{K_v}/K_v)$ and let $\rho_{y,v}$ be the restriction of $\rho_y$ to $G_{K_v}$. The task is to prove that the map sending $y \in Y(K_v)$ to the isomorphism class of $\rho_y^{\mathrm{ss}}|_{G_{K_v}}$ has finite fibres. It is enough to consider one residue disk at a time, say the residue disk of $y_0$. Using $p$-adic Hodge theory, we obtain a period map into the de Rham cohomology $\mathrm{H}^i_{\mathrm{dR}}(X_y/K_v)$ with its Hodge filtration and a semilinear Frobenius $\phi$. Using the Gauss–Manin connection we obtain a map from the residue disk of $y_0$ in $Y(K_v)$ to the set of $K_v$-points of a certain flag variety $\mathcal{F}$ of subspaces of $\mathrm{H}^i_{\mathrm{dR}}(X_{y_0}/K_v) = \mathrm{H}^i_{\mathrm{crys}}(X_{y_0}) \otimes K_v$. One needs to prove that the period map is injective and that its image has finite intersection with an orbit of the centraliser $Z(\phi)$ of $\phi$, which involves proving that $Z(\phi)$ is not too large. It remains then to circumvent the problem that $\rho_y$ can fail to be semisimple.

**Plan of talks**

9 October

(a) **Introduction:** Alexei Skorobogatov

Give an overview of the paper [LV]. Also discuss Chabauty's method, Kim's non-abelian generalisation of Chabauty's method, and compare to the methods of [LV].

(b) **Faltings's proof:** Ambrus Pal

Sketch Faltings's proof of the Mordell conjecture. Discuss in detail Faltings's finiteness result on the semisimplification of the Galois representations seen in the étale cohomology of the fibers of a proper smooth family, as stated in [LV, Lemma 2.3].

16 October

(a) **De Rham cohomology over $\mathbb{C}$:** Richard Thomas

Recall de Rham cohomology, its comparison with singular cohomology, and the monodromy representation. Introduce the Gauss-Manin connection and explain how it can be used to identify the de Rham cohomology groups of two fibers that are sufficiently close to each other. Discuss the complex period morphism. Give an example, such as a family of elliptic curves.

(b) **de Rham cohomology over arbitrary fields [KO],[LV, §3.3]:** Lorenzo La Porta

Consider a smooth morphism $X \to Y$ of smooth, separated schemes of finite type over a field of characteristic 0. Recall algebraic de Rham cohomology and the Gauss–Manin connection in this setting. Prove that the Gauss–Manin connection is a flat connection, and explain how it can be used to identify the de Rham cohomologies of nearby fibers. End by stating Diagram (3.9) of [LV].

23 October

(a) **Introduction to crystalline cohomology [ChL]:** Ambrus Pal

Introduce crystalline cohomology. State the comparison with $p$-adic étale cohomology and with de Rham cohomology. Give examples.

(b) **$p$-adic Hodge theory and period maps [LV, §3.4, 3.5]:** Ana Caraiani

Discuss crystalline Galois representations and the fully faithful functor to filtered $\phi$-modules. Introduce the $p$-adic period morphism and discuss the comparison with the complex period morphism. State and prove Proposition 3.3 of [LV].

30 October

(a) **The $S$-unit equation [LV, §4]:** Domenico Valloni

Explain the proof of the fact that the $S$-unit equation has finitely many solutions, using a modification of the Legendre family.

(b) **Outline of the proof of Mordell [LV, §5]:** Alex Torzewski

Introduce the notion of an abelian-by-finite family and define the notion of size. Explain the argument for the proof of Mordell's conjecture [LV, §5], including stating without proof Proposition 5.3 of *loc.cit.* and stating the key properties of the Kodaira–Parshin family.

6 November

(a) **Proof of Proposition 5.3 [LV, §6]:** Domenico Valloni

Recall Proposition 5.3 of [LV] and explain its proof. Refer to §2 of [LV] as needed to handle the possible failure of semisimplicity.

(b) **Kodaira–Parshin families [LV, §7]:** Pedro Lemos

Explain the construction of Kodaira–Parshin families.

13 November

(a) **Monodromy of Kodaira–Parshin families [LV, §8], I**: Andrew Graham

(b) **Monodromy of Kodaira–Parshin families [LV, §8], II** Pol van Hoften

20 November

(a) **The Bakker–Tsimerman theorem [BT]:** Gregorio Baldi

In preparation for discussing the higher-dimensional case, discuss the Bakker–Tsimerman theorem as well as its Corollary 9.2 of [LV].

(b) **The $p$-adic variant of Ax–Shanuel [LV, Lemma 9.3]:** Rachel Newton

27 November

(a) **The higher-dimensional case:** Matteo Tamiozzo

State the main result in the higher-dimensional case, namely [LV, Thm. 10.1] and sketch its proof, except the proofs of Lemmas 10.4 and 10.5 that will be discussed later.

(b) **Application to hypersurfaces [LV, §10.2]:** James Newton

Explain the application of the main theorem in the higher-dimensional case to the moduli of hypersurfaces of degree $d$ in $\mathbb{P}^n$, assuming $n$ is large enough and $d$ is larger than some bound depending on $n$.

4 December

(a) **Proof of [LV, Lemma 10.4]:** Alexei Skorobogatov

(b) **Proof of [LV, Lemma 10.5]:** Alexei Skorobogatov

11 December

(a) **Proof of Prop. 10.6 [LV, §11], I:** TBA

(b) **Proof of Prop. 10.6 [LV, §11], II:** TBA

# References

[BT] B. Bakker and J. Tsimerman. The Ax-Schanuel conjecture for variations of Hodge structures. Invent. Math. 217 (2019), no. 1, 77–94.

[ChL] A. Chambert-Loir. Cohomologie cristalline : survol. https://webusers.imj-prg.fr/ antoine.chambert-loir/publications/papers/cristal.pdf

[KO] N.M. Katz and T. Oda. On the differentiation of de Rham cohomology classes with respect to parameters. J. Math. Kyoto Univ. 8 1968 199–213.

[LV] B. Lawrence and A. Venkatesh. Diophantine problems and $p$-adic period mappings. arXiv:1807.02721

[Poonen] B. Poonen. $p$-adic approaches to rational and integral points on curves. http://www-math.mit.edu/ poonen/papers/p-adic_approach.pdf