LONDON NUMBER THEORY STUDY GROUP: FARGUES'S CONJECTURE

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This study group will be focused on a recent development in the Langlands program, namely Fargues's conjecture on the geometrization of the local Langlands correspondence. This is inspired by ideas from the geometric Langlands program, but a striking aspect is that the conjecture makes sense both over $E = \mathbb{F}_q((t))$ and over finite extensions $E/\mathbb{Q}_p$. The goals of the study group are to define the geometric objects that come into the statement of the conjecture (adic spaces, perfectoid spaces, diamonds, the Fargues-Fontaine curve), to go through Fargues's new proof of local class field theory which is essentially the GL_1 case of the conjecture, and to state Fargues's conjecture precisely.

Below, we let E to be a local field with residue field $\mathbb{F}_q$.

Talks

Note that the order (and precise contents) of the talks from November 15th onwards is provisional, and may change depending on the availability of speakers.

Talk 1: Introduction to Fargues's conjecture (90 mins). October 4 2017. Speaker: Toby Gee

Give context, motivation and an overview for Fargues's conjecture. Introduce the classical local Langlands correspondence over p-adic fields, with a focus on what is known for GL_n. Present the geometric proof of class field theory over function fields, in the unramified case. Describe Fargues's conjecture and give an overview of the rest of the study group.

References: [4], [5], Chapter 2 of [13].

Talk 2: Adic spaces (90 mins). October 4 2017. Speaker: Ben Heuer

Introduce Huber pairs, affinoid adic spaces and general adic spaces. Describe rational subsets and the structure (pre-)sheaves. Discuss the relationship to rigid analytic spaces and formal schemes. Give examples. Draw a picture of

$$\mathrm{Spa}\left(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle\right)$$

and discuss the 5 different types of points on it.

References: Section 2 of [10] and the references mentioned there.

Talk 3: Perfectoid spaces (2 talks of 90 mins each). October 11 2017. Speaker: Kevin Buzzard

Part 1: Define perfectoid fields, give examples in both mixed and equal characteristic, explain the tilting construction for perfectoid fields and the theorem of Fontaine-Wintenberger. Define perfectoid algebras over a perfectoid field and, more generally, perfectoid spaces. Give examples of perfectoid spaces.

Part 2: Discuss the étale site of a perfectoid space and the almost purity theorem. Sketch the proof of the tilting equivalence. Also define the category $\mathrm{Perf}_{\mathbb{F}_q}$ of perfectoid spaces over $\mathbb{F}_q$ and the pro-étale topology on it.

References: [10].

Talk 4: The Fargues-Fontaine curve (2 talks of 90 mins each). October 18 2017. Speakers: Tibor Backhausz and Raffael Singer

Part 1: Introduce the Fargues-Fontaine curve $X = X_{F,E}$ over a perfectoid field F over $\mathbb{F}_q$. Define both its schematic and adic incarnations. Discuss the relationship of the curve with "untilts" of F. Describe how to go from isocrystals to vector bundles over the curve, and state the classification of vector bundles. (You might also want to state the GAGA theorem that vector bundles over the schematic curve are in equivalence with vector bundles over the curve.)

Part 2: Introduce the relative Fargues-Fontaine curve $X_{S,E}$ over a perfectoid space S over $\mathbb{F}_q$. Describe the relationship of the relative Fargues-Fontaine curve to untilts of S. Introduce the relative Robba ring. Describe vector bundles over the Fargues-Fontaine curve in terms of φ-modules over the Robba ring.

References: [2], [3], [7].

Talk 5: The classification of vector bundles on the Fargues-Fontaine curve (2 talks of 90 mins each). October 25 2017. Speakers: Rebecca Bellovin and Carl Wang-Erickson

Part 1: Recall the statement of the classification theorem for vector bundles on the Fargues-Fontaine curve, in the case when F is algebraically closed. Give an outline of the proof. Prove Proposition 6.13 of [3]. Discuss modifications of vector bundles associated to p-divisible groups, the case of Hodge-de Rham periods. Discuss the fact that for Lubin-Tate spaces the weakly admissible and admissible loci agree and translate this into Theorem 6.20 of [3].

Part 2: Discuss modifications of vector bundles associated to p-divisible groups, the case of Hodge-Tate periods, proving Theorem 6.27 of [3]. Finish the proof of the classification theorem. Discuss Galois descent of vector bundles in the case when F is not necessarily algebraically closed.

References: Section 6 of [3], [2]

Talk 6: Banach-Colmez spaces (180 mins). November 1 2017. Speaker: Arthur-César Le Bras

Define the category of Banach-Colmez spaces following [8]. Recall the universal cover of a p-divisible group from [11] and explain why this gives rise to a Banach-Colmez space. Prove that the global sections functor induces an equivalence of categories between a certain abelian subcategory $\mathrm{Coh}_X^{-,-}$ of the derived category of coherent sheaves on the Fargues-Fontaine curve and the category of Banach-Colmez spaces.

References: [11], [8]

Talk 7: The classification of G-bundles on the Fargues-Fontaine curve (90 mins). November 8 2017. Speaker: Johannes Anschütz

When E is a non-archimedean local field and G/E is a connected reductive group, discuss the category of isocrystals with G-structure and the Kottwitz set $B(G)$. Prove the classification of G-bundles on the Fargues-Fontaine curve, following [1].

References: [6], [9], [1].

Talk 8: Diamonds (90 mins). November 8 2017. Speaker: Christian Johansson

Introduce the category of diamonds. Give examples, such as the diamond incarnation of the Fargues-Fontaine curve. Define the underlying topological space of a diamond and introduce (locally) spatial diamonds. Discuss the functor from analytic adic spaces over $\mathbb{Z}_p$ to diamonds.

References: [12].

Talk 9: Where does the curve and the conjecture come from? What happened in Trieste, Orsay and Berkeley? (180 mins). November 15 2017. Speaker: Laurent Fargues

Talk 10: New proof of local class field theory in characteristic p (2x90 mins). November 22 2017. Speakers: Robin Bartlett and David Helm

Talk 11: New proof of local class field theory in characteristic 0 (90 mins). November 29 2017. Speaker: James Newton

Talk 12: The statement of Fargues's conjecture I (90 mins). November 29 2017. Speaker: Ana Caraiani

Talk 13: The statement of Fargues's conjecture II (90 mins). December 6 2017. Speaker: Ana Caraiani

Talk 14: Character sheaves and Fargues's conjecture (90 mins). December 6 2017. Speaker: Ian Grojnowski

Talk 15: Fargues's conjecture for GL_n and the cohomology of the Lubin-Tate and Drinfeld towers (180 mins). December 13 2017. Speaker: Andrea Dotto

References

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