ma414soln5.tex

MA414 SOLUTIONS 5. 13.2.2012

Q1. *Proof (Doob's Submartingale Inequality).* Let

$$F := \{\max_{k \le n} X_k \ge c\}, \quad F_k := \{X_0 < c\} \cap \{X_1 < c\} \cap \ldots \{X_{k-1} < c\} \cap \{X_k \ge c\}.$$

Then F is the disjoint union $F = F_0 \cup \ldots \cup F_n$. Also $F_k \in \mathcal{F}_k$, and $X_k \ge c$ on F_k. So

$$\begin{aligned} E[X_n I(F_k)] &\ge E[X_k I(F_k)] \qquad (X \text{ a submg}) \\ &\ge cE[I(F_k)] = P(F_k). \end{aligned}$$

Sum over k:

$$\begin{aligned} E[X_n] \ge E[X_n I(F)] &= \sum_k E[X_n I(F_k)] \\ &\ge \sum_k cP(F_k) \\ &= cP(F). \qquad // \end{aligned}$$

Q2 *(Second Borel-Cantelli Lemma for Pairwise Independence).* If the events A_n are pairwise independent, then $\sum P(A_n)$ diverges implies $P(\limsup A_n) = P(A_n \ i.o.) = 1$.

Proof. For A_n pairwise independent, put $S_n := \sum_1^n I(A_i)$, $S := \sum_1^\infty I(A_i)$, $m_n := E[S_n] = \sum_1^n P(A_i)$.

$$var(S_n) = E[(S_n - m_n)^2] = E[(\sum_{i=1}^n (I(A_i) - EI(A_i))(\sum_{j=1}^n (I(A_j) - EI(A_j))]$$

$$= E[\sum_i \sum_j (\ldots)(\ldots)] = \sum_i E[(\ldots)^2] + \sum_{i \ne j} E(\ldots)(\ldots)] = \sum_i E[(\ldots)^2]$$

(the sum over $i \ne j$ is 0, as there by pairwise independence and the Multiplication Theorem $E[(\ldots)(\ldots)] = E[(\ldots)]E[(\ldots)] = 0.0 = 0$ – variance of sum = sum of variances under pairwise independence). As $I(A_i)$ is Bernoulli with parameter $P(A_i)$, its variance is $P(A_i)[1 - P(A_i)] \le P(A_i)$. So

$$E[(S_n - m_n)^2] \le \sum_1^n P(A_i) = m_n,$$

which increases to $+\infty$ as $\sum P(A_n)$ diverges, by assumption. But

$$\begin{aligned}
P(S \leq m_n/2) &\leq P(S_n \leq m_n/2) \qquad (S_n \leq S) \\
&= P(S_n - m_n \leq -m_n/2) \\
&\leq P(|S_n - m_n| \geq m_n/2) \\
&\leq \frac{4}{m_n^2} var(S_n) \qquad \text{(by Tchebycheff's Inequality)} \\
&\leq 4/m_n \qquad \text{(by above)} \\
&\to 0 \qquad (n \to \infty).
\end{aligned}$$

But the LHS increases to $P(S < \infty)$, by continuity (= σ-additivity) of $P(.)$. So $P(S < \infty) = 0$: $P(\sum I(A_n) < \infty) = 0$, i.e. $P(\sum I(A_n) = \infty) = 1$. This says that $P(A_n \; i.o.) = 1$: $P(\limsup A_n) = 1$. //

Q3. (i) For $s < t$, $M_s = E[M_t|\mathcal{F}_s]$ as M is a mg. So by the conditional Jensen inequality,

$$\phi(M_s) = \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s],$$

which says that $\phi(M)$ is a submg.
(ii) If M is a submg, $M_s \leq E[M_t|\mathcal{F}_s]$. As ϕ is non-decreasing on the range of M,

$$\phi(M_s) \leq \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s]$$

(the second inequality by conditional Jensen as above), and again $\phi(M)$ is a submg.

Q4. For $S_n = \sum_1^n X_k$ with X_k independent with mean 0, $S = (S_n)$ is a mg. As $\phi(x) := x^2$ is convex, S^2 is a submg, and $E[S_n^2] = varS_n = \sum_1^n varX_k$. By Doob's Submg Inequality,

$$P(\max_{k \leq n} |S_k| \geq c) = P(\max_{k \leq n} S_k^2 \geq c^2) \leq E[S_n^2]/c^2 = c^{-2} varS_n = c^{-2} \sum_1^n varX_k,$$

giving Kolmogorov's Inequality. //

NHB