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DOOB: A HALF-CENTURY ON

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Abstract

Probability theory, and its dynamic aspect stochastic process theory, is both a venerable subject in that its roots go back to the mid-seventeenth century, and a young one in that its modern formulation is comparatively recent – well within living memory. The year 2003 marks the seventieth anniversary of Kolmogorov's *Grundbegriffe*, usually regarded as inaugurating modern – measure-theoretic – probability theory (for historical background, see e.g. Bingham (2000)). It also marks the fiftieth anniversary of Doob's *Stochastic Processes* ('Doob' below, unless it is clear from context that we refer to J. L. Doob the person). The profound and continuing influence of this classic work prompts the present piece.

1 Pre-Doob

Before the emergence of measure theory, the machinery needed to handle probability rigorously did not exist. Hardly surprisingly, the subject was not regarded as mathematically respectable by pure mathematicians. The wonder is that so much of lasting value was achieved – witness the books, and work, of Markov and Poincaré, for example. Richard von Mises, writing in the 1920s, regarded probability not only as not being fully mathematical, but as not being fully mathematicisable.

This view had begun to change even before the *Grundbegriffe*. Paul Lévy (1886-1971) published the first measure-theoretic book on probability, *Calcul des Probabilités*, in 1925, following this with his two enduring classics, *Théorie de l'Addition des Variables Aléatoires* in 1937 and *Processus Stochastiques et Mouvement Brownien* in 1948. In his papers, Lévy made profound contributions, not only to limit theorems – the enduring theme of his life's work was the central limit problem and in particular, limiting Gaussianity, in the most natural and general setting – but also to the Lévy-Khintchine formula and to what are now called Lévy processes. See the obituary Loève

(1973) for background on Lévy's life and work. One might add that Lévy's writing style contrasts strongly with Doob's. Lévy is intuitive ('Alors, ces probabilités sont assez petites, ...'), and discursive (the discussion ends with what has been proved, in italics – the style of French textbooks of the time, pre-Bourbaki). Doob is careful, formal – theorem followed by proof, in the Landau Satz-Beweis style – and precise.

Partly at the instigation of G. H. Hardy (1877-1947), Harald Cramér (1893-1985) wrote his influential Cambridge Tract, Cramér (1937). With his background in pure mathematics (originally analytic number theory), Cramér was able to take up the lead of the *Grundbegriffe* and present, at greater length, a synthesis of much of the subject as it then stood. The title of the present piece is in part a tribute to Cramér (1976).

Also coming to probability from a background in analysis was William Feller (1906-1970). Feller began to write on probability in 1935. His deepest work was on the interplay between Markov processes on the one hand and second-order linear differential operators on the other – in particular, questions of classification and boundary conditions. The first edition of his famous book 'Feller Volume 1' appeared in 1950, only three years before Doob, and it is interesting and instructive to compare the two.

The spectacular career of Wolfgang Döblin (1915-1940), a tragic early victim of World War II, made a profound impact on the subject (and on Doob's book) in the last three years of his life. For background on the man who ended his life as Vincent Doeblin, see e.g. Lindvall (1991), Cohn (1993). Other important influences were the work of the Russian school (Kolmogorov, Khintchine and others), the Polish (Marcinkiewicz, Zygmund, ...), the Japanese (Itô, ...) and the American (Wiener, Stone, ...).

Before embarking on a discussion of Doob, it might be as well to comment on the influence, pre-Doob, of Norbert Wiener (1894-1964). Wiener's greatest contribution to probability and stochastic processes was his 1923 construction of Wiener measure – essentially, the rigorous proof that 'Brownian motion is continuous'. But in his writings of the early 1930s - on Fourier integrals, generalized harmonic analysis, Tauberian theory etc. (and in the 1940s on time series, particularly spectral aspects, and prediction), Wiener convincingly demonstrated the power of Fourier analysis, rigorously and solidly based on the Lebesgue integral, Lebesgue measure and measure theory. This played an important role in convincing the mathematical public – applied workers as well as theoreticians – that one had to make the investment of learning measure theory. The resulting sea-change in attitude within the profession helped to prepare the ground for Doob.

Doob's book was in fact planned to be a collaborative work with Wiener. To quote the penultimate paragraph of the Preface:

'Chapter XII, on prediction theory, is somewhat out of place in the book, since it discusses a rather specialized problem. It was put in because of the importance of the subject matter, and because of the lack of material on prediction theory, in the usual language of probability, readily available to the American reader. I had the benefit of stimulating conversations with Norbert Wiener on this subject'.
For background on Wiener, see e.g. his *Collected Works* (four volumes), the biography Masani (1990), and the review of this, Doob (1990).

2 Doob

Joseph Leo Doob (1910-) was – like several of the probabilists already mentioned – originally a pure mathematician and analyst. His early interests lay in complex analysis and potential theory. Confronted as a young man by the great difficulty of finding an academic job during the Great Depression and its aftermath, he took the advice of Harold Hotelling and decided to specialize in probability and statistics. His book cites 13 of his papers, written in the period 1934-1951.

The unchallengeable nature of measure theory as the essential mathematical language for probability, and Doob's forthright attitudes and style, are both well exemplified by two sentences from the third paragraph of the Preface to Doob:

'There has been no compromise with the mathematics of probability. Probability is simply a branch of measure theory, with its own special emphasis and field of application, and no attempt has been made to sugar-coat this fact.'

The many-sided nature of probability and stochastics – as between pure and applied aspects, as between mathematics and statistics, etc. – is also well exemplified by a sentence from Doob's next paragraph:

'There is probably no mathematical subject which shares with probability the features that on the one hand many of its most elementary theorems are based on rather deep mathematics, and that on the other hand many of the most advanced theorems are known and understood by many (statisticians and others) without the mathematical background to understand their proofs.'

Inescapably, the first problem facing the author of a book on probability or stochastic processes remains what it was half a century ago – how to handle measure theory. (The present writer met an echo of this when examining recently in the University of Cambridge. Both measure theory and stochastic processes are in the curriculum - *but*, the numbers taking the first are tiny compared to those taking the second.) There are basically three ways one can proceed. Firstly, one can present a non-measure theoretic treatment as best one can. Distinguished classics of this kind include Feller (1950/57/68), Bartlett (1955/66/78) – originally published two years after Doob but conceived (with J. E. Moyal) in 1946 – and, later, Karlin and Taylor (1975), (1981), Grimmett and Stirzaker (1982/92/2001). Second, one can expound measure theory and probability together (or consecutively in the same book), as in, e.g., Kingman and Taylor (1966) or Billingsley (1979/86/95). Third, one can proceed as Doob does, and assume measure theory as known to the reader, referring to a standard book on measure theory as needed. Doob refers to Halmos (1950) (Paul R. Halmos was a pupil of Doob's) – but summarises what he needs of measure theory in a 24-page Supplement at the end of the book.

The next basic problem concerns the gap between discrete and continuous time. (Readers may care to ask themselves whether time is discrete or continuous. One might reply that how one perceives or experiences time depends on whether one uses a digital watch or one with hands. And of course, how one perceives time affects how one models it.) The essence of measure theory lies in the property of *countable additivity*. The countability endemic here sits comfortably with the countable setting of discrete time. (The time-set may well be *finite*, with the setting still having the character of a stochastic process problem rather than one in enumerative combinatorics. This happens in, e.g., the binomial tree model of mathematical finance, for which see e.g. Bingham and Kiesel (1998/2004).) But countable additivity does *not* sit comfortably with the uncountable time-set encountered in continuous time. Stochastic process theory – the dynamic side of probability, or the mathematics of randomness unfolding with time – will thus, inescapably, always be a subject presenting some difficulty to those who learn it, teach it or write about it.

Doob begins (Chapter I: Introduction and probability background, 45p.) with a summary of what he needs from probability. Here he treats the

Daniell- Kolmogorov theorem (as in the *Grundbegriffe*) – essentially the existence theorem for a stochastic process, given the minimal raw material of an appropriately consistent set of finite-dimensional distributions. The main emphasis is on conditioning (Kolmogorov's treatment of conditioning in the *Grundbegriffe*, using the Radon-Nikodým theorem, being by now well established). He closes with a treatment of characteristic functions (including – with an eye on his intended treatment of infinite divisibility – more on inequalities for them than most readers nowadays will know).

Doob continues with Chapter II: Definition of a stochastic process – Principal classes, 56p. The starting-point (§1) is a consistent set of finite-dimensional distributions, and the Kolmogorov construction from it of a stochastic process (basically, *one* infinite-dimensional object unifying *infinitely many* consistent finite-dimensional objects). He addresses (§2) the 'pitfall of uncountability' (above): essentially, that, with an uncountable index set (in continuous time, say), the sample paths of the process may lack enough regularity to be tractable. He addresses this with one of his key technical contributions – his theory of *separability, measurability* and *versions.* (In brief – p.66 – separability is not a restriction on finite-dimensional distributions, measurability is.) The rest of the chapter is a brief introduction to the particular classes of stochastic process to which the remaining ten chapters will be devoted. In particular, the Gaussian section contains (p.78) the intriguing statement '... very few facts specifically true of Gaussian processes are known'.

Chapter III (Processes with mutually independent random variables, 46p.) addresses the core of probability theory, as would be expounded nowadays in a book on probability theory at measure-theoretic level but without a specific focus on stochastic processes. It begins with zero-one laws. It then treats random series (including, for series with independent terms, the equivalence of convergence almost-surely, in probability and in distribution). It proves Kolmogorov's strong law of large numbers. It treats infinite divisibility, and derives the Lévy-Khintchine formula.

In Chapter IV (Processes with mutually uncorrelated or orthogonal random variables, 22p.), the restriction to probability (or even finite) measures is dropped. Versions in this degree of generality of the law of large numbers and results on random series are given. So too – with a view to applications to time series later – are results on factorization of spectral densities. Doob emphasises that the natural setting here is that of general orthogonal series, and accordingly refers to analysis books such as Kaczmarz and Steinhaus

(1935/51), Stone (1932) and Zygmund (1935/59) for proofs.

Doob turns in Chapter IV (Markov processes – Discrete parameter, 64p.) to Markov chains. Finite chains are treated in some detail. Thus, one cannot remain in transient states forever (and so, not all states can be transient). But, persistent (or recurrent) states are not broken down into null and positive, as in Feller, and so the non-existence of null states in finite chains is not given. Card-shuffling is discussed (the modern reader will recall Diaconis' result, *seven shuffles suffice*). General state spaces (§5, p.190-218) are treated in one go, with heroic disregard for the frailties of the reader. Topics include Döblin's condition and exponential convergence. Finally, the law of large numbers and central limit theorem are proved, using 'Döblin's trick' (cf. Chung (1960)).

Markov processes with continuous parameter follow in Chapter VI (57p.). Finite chains are treated: the Q-matrix, holding times, step-function nature of the sample paths, etc. With continuous state space, Döblin's condition is used; the sample paths may have discontinuities worse than jumps. Diffusions are treated in the light of Itô's work (1944-1951) on stochastic integrals and stochastic differential equations.

Doob's Chapter VII on martingales (98p.) is the most famous in the book. Because it has been so influential, it still has a modern feel to it, apart from terminology (thus filtrations are everywhere, but not by name; our submartingales and supermartingales appear here as semi-martingales and lower semi-martingales, etc.). After the basic definitions (§1), Doob turns (§2) to games of chance, gambling systems and the Optional Stopping Theorem. In §3 (Fundamental inequalities), one finds Doob's maximal inequality (generalizing Kolmogorov's, itself generalizing Chebyshev's) and the upcrossing inequality. These are applied to convergence theorems in §4 (uniformly integrable martingales; martingale convergence; martingale version of Borel-Cantelli; reversed martingales). Sums of independent terms are treated in §5 (random series) and §6 (strong law). Integration and differentiation follow in §§7,8, likelihood ratios and sequential analysis in §§9,10. Continuous-parameter martingales are treated in §11 (which, at 36 pages, again makes demands on the reader's stamina). In particular, regularization of paths is covered, and Lévy's martingale characterization of Brownian motion is given. The chapter closes (§12) with applications of martingale theory to sample-path continuity (a sample result of Lévy: sample paths of Lévy processes have at worst jump discontinuities).

Chapter VIII (Processes with independent increments, 35p.) is devoted to (in modern terminology) Lévy processes. Processes with orthogonal increments follow in Chapter IX (27p.). Stochastic integrals are discussed, from the point of view of Wiener and Stone. Application is made to Campbell's theorem (shot-noise processes), and to a version of the representation theorem for Brownian martingales.

Stationary processes are treated in discrete time (Chapter X, 55p.) and continuous (Chapter XI, 53p.). Here one finds the strong law of large numbers, Bochner's theorem, and the mathematics needed for time series in the frequency domain (spectral representation, Wold decomposition, ...). Linear filtering is also discussed (XI.9).

The book ends with the admittedly more specialized Chapter XII (Linear least squares prediction – Stationary (wide sense) processes, 38p.). There is a thorough discussion of the 'Kolmogorov-Wiener filter' – linear prediction given the entire past, in the square-integrable case – the Wold decomposition again playing a central role. Both Kolmogorov and Wiener were motivated by wartime problems of fire control, particularly against aircraft. One source here is Wiener's 'Yellow Peril' (Wiener (1949) – based on a version of 1942 with circulation restricted by wartime security). Another is Doob's 40-page Berkeley Symposium paper, Doob (1949).

3 Post-Doob

There is general agreement that the two enduring classics of their time are Feller Volume 1 and Doob. I recall 'Kingman's dictum' from the sixties and seventies – 'It's all in Doob' (just as a modern stochastic analyst might say, 'All you need is Itô's lemma' – a dictum I learned from Michael Harrison).

The most obvious successor to Doob, in terms of scope and aims, was perhaps Loève (1955/60/63). Revealingly, Loève's book split into two volumes (1977, 1978) for later editions. Similarly, Gihman and Skorohod (1969) gave rise to a three-volume work. As the field continued to develop, it became clear to authors that one could no longer aim to cover everything in stochastic processes in one book of reasonable size. Accordingly, the literature ramified, and books on particular kinds of process or particular aspects of the field began to appear. It is interesting to observe the extent to which Doob's book set the research – and textbook – agenda.

Some aspects of stochastic processes are covered in any standard text on probability. Of many such, we mention first Feller Volume 2 (Feller

(1966/71)). This influential text was planned as part of a three-volume work, with Volume 3 on stochastic processes. Feller's death in 1970 prevented Volume 3 being written. Feller returned repeatedly to what he had already written, revising and polishing. Doob was content to let his book speak for itself, and eventually 'die quietly on the shelf' (personal remark to the author, 1975-6). Later probability books of comparable level include Breiman (1968), Chung (1968/74/2001) and Billingsley (1979/86/95).

For Markov chains, the standard specialist work remains Chung (1960 /67). For Markov processes, the next generation is dominated by Dynkin (1965), a two-volume work. The role of Döblin's condition is reflected in a number of variants and alternatives, under which Markov processes on general state- spaces may be handled, to some extent, in the manner of Markov chains. The best-known such condition is Harris recurrence. For monograph accounts, see Nummelin (1984), Revuz (1984), Meyn and Tweedie (1993), The role of exponentially fast convergence led to the study of *geometric ergodicity*, by Kendall, Vere-Jones, Pitman, Tweedie and others.

The theory of infinite divisibility has always been intimately linked with that of processes with (stationary) independent increments. These were previously studied under a variety of unwieldy names, but were very properly called *Lévy processes* since at least the early seventies. The year after Doob appeared, Gnedenko and Kolmogorov (1954) was published in English. A good deal of this material appears in Fellerian style in Feller Volume 2 (1966/71). Lévy processes deserve and get a chapter in any serious text on stochastic processes, but received two fine monograph treatments in Bertoin (1996) and Sato (1999) – the first more probabilistic, the second more analytic.

Laws of large numbers received a monograph treatment in Révész (1968), ergodic theorems in Krengel (1985), random series in Kahane (1985) and random walks in Spitzer (1964).

Gaussian processes received a great deal of attention, in the Russian school, the French (X. Fernique), the American (S. M. Berman, M. B. Marcus) and the Japanese. The textbook literature includes Ibragimov and Rozanov (1978). For further references, developments involving abstract Wiener space, white-noise calculus etc., see Janson (1997).

The literature on time series has exploded in the last half-century. An early post-Doob book is Whittle (1963). The Kolmogorov-Wiener filter was supplemented by the Kalman filter (just in time for its deployment in its first natural field of application - control of manned spacecraft in real time). For a treatment of filtering and control. see e.g. Davis (1977). For a contemporary view of time series, see e.g. Brockwell and Davis (1987) and the references cited there.

Martingales have become ubiquitous since Doob – we live in the age of martingales. They have been studied particularly intensively by D. L. Burkholder, a colleague of Doob's at Illinois. For a fine textbook account, see Neveu (1975). For martingale central limit theory, see Hall and Heyde (1980). Note that Doob covers central limit theory for the Markov-chain but not the martingale case.

Diffusions in one dimension may be treated by a variety of methods. See e.g. Itô and McKean (1965), Breiman (1968). In the multi-dimensional case, there is essentially only one approach, the Stroock-Varadhan approach via martingale problems (Stroock-Varadhan (1979)). For an excellent treatment of much of the material duscussed above, see the two-volume work Rogers and Williams (1994), (1987).

One of the main achievements of Paul-André Meyer (1934-2003) was to develop (particularly following the work of Kunita and Watanabe of 1967 on square-integrable martingales) the theory of stochastic integration; see e.g. Meyer (1976). Meyer's formulation involves *predictable* integrands and *semi-martingale* integrators (semi-martingales in Meyer's sense, not Doob's!). For textbook accounts of the related theories of stochastic integrals and stochastic differential equations, see e.g. McKean (1969), Ikeda and Watanabe (1981), Øksendal (1985), Protter (1990), Karatzas and Shreve (1987), Revuz and Yor (1991). A different approach, via 'rough paths', has recently been given by Lyons and Qian (2002).

One topic not covered in Doob (perhaps surprisingly, in view of Doob's background in complex analysis and potential theory) is the link between potential theory and probability – in particular, with Markov processes. This goes back to Kakutani (1944), who showed that classical (Newtonian) potential theory is intimately linked with (corresponds to, one might say) Brownian motion. It was realised – most notably by Hunt, in a series of papers in 1957-58 – that one could usefully associate potential theories to Markov processes, and much work was done – by Brelot, Deny and others – on axiomatization and generalization of potential theory. The book *Probability and Potential* (Meyer (1966)) explored this link – and was later re-written in a five-volume work of the same title by Dellacherie and Meyer. Soon after Meyer's book, Blumenthal and Getoor (1968) appeared – for long the standard work on Markov processes and potential theory. Doob's interest in potential theory resulted in his later – and even longer – book, Doob (1984), written in his retirement.

The other principal achievement of Meyer and the Strasbourg (more generally, French) school has been their 'general theory of processes'. A monograph treatment of Markov process theory in the light of this general theory of processes was given by Sharpe (1988). For an insight into the formidable technical problems in this area, see the review Rogers (1989). One service done by the book Protter (1990) was to make available for a wider audience – including in particular applied workers – what the 'probabilist in the street' needs to know about the general theory of processes.

As a glance at the length of the list of books cited here underlines, there will never be another Doob-the-book. If one person deserves to stand comparison with Doob-the-man during the half-century discussed here, that person must be Meyer. It is perhaps fair to remark that while the broad thrust of Meyer's work began with Markov processes, it changed towards martingales during his lifetime, and it is here, it seems, that his impact will be deepest.

4 Post-Doob: the legacy to applied probability

The viewpoint and results of martingale theory are perhaps the most important of the legacies of Doob's book. Their influence is all around us. In mainstream probability theory, see e.g. Rogers & Williams (1994), (1987); in limit theorems, see Ethier & Kurtz (1986) (martingale problems), Jacod & Shiryaev (1987) (convergence of semi-martingale characteristics). In analysis, see Durrett (1984). In statistics, see e.g. the book of Heyde (1997) on quasi-likelihood, and the extensive literature on sequential analysis. A further area of statistics where martingale methods are crucial is survival analysis and event-history data; see e.g. Andersen *et al.* (1993).

Within applied probability, one area where martingales have been particularly crucial is queues. Here, interest centres on random discrete times, when customers arrive and depart, etc., and the relevant theory – 'Palm-martingale calculus' – is a discrete, Poisson-based counterpart to the (harder, better-known and earlier) continuous, Gaussian-based Itô calculus. For textbook expositions, see e.g. Brémaud (1981), *Point processes and queues: Martin-*

gale dynamics, Baccelli & Brémaud (1994), *Elements of queueing theory: Palm-martingale calculus and stochastic recurrences*, and Brémaud (1999).

Rather in the same vein is applications to point processes. Here the main theme is *intensity* – the propensity of what can happen, to happen; typical happenings here are such things as earthquakes and volcanic eruptions. For background, see e.g. Daley & Vere-Jones (1988), esp. Ch. 12-13.

Branching processes provide another rich field of application. See for example Athreya & Ney (1972), Jagers (1975), Lynch (2000) and references there, and many papers by J. D. Biggins. Similarly for other population models such as birth-and-death processes, and within mathematical biology more generally.

The theory of collective risk has been central in applied probability since the early work of Cramér. Random-walk methods have long been used here; see e.g. Feller Volume 2. Martingale methods were fruitfully introduced into the area of insurance and actuarial mathematics in the 1970s by H. U. Gerber; see e.g. Gerber (1986).

Finally, following the work of Black and Scholes in 1973, the field of mathematical finance has been transformed by the introduction of relevant stochastic methods. First, Itô calculus was introduced into the field by Merton as early as 1973. Second, martingale methods were introduced by Harrison & Pliska (1981). The crux of the subject is *equivalent martingale measures*: probability measures equivalent to the original one under which discounted prices become martingales. For background and references, we refer to Bingham & Kiesel (1998/2004).

5 Postscript

The name Doob is so famous, and so unusual, that its origins may be worth recording here, for historical interest. Doob's father was Czech, and 'dub' in Czech means 'oak'. (Readers may recall Alexander Dubcek and the Prague Spring of 1968. 'Dubcek' means 'little oak' – to the delight of the cartoonists of that time.) When Doob's father, then Dub, emigrated to the USA, he got sick of being called 'dub', and changed his name to Doob. To change the spelling to preserve the pronunciation is rare. Usually, the spelling is preserved and the pronunciation changed (thus the distinguished probabilist and statistician Jack Wolfowitz – father of the Deputy Secretary of Defense – chose to be 'wolf-oh-wits'; similarly for Wiener, etc.).

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References

Andersen, P. K., Ø. Borgan, R. D. Gill and N. Keiding (1993). *Statistical models based on counting processes*. Springer, New York.

Athreya, K. B. and Ney, P. E. (1972). *Branching processes*. Springer, New York.

Bacelli, P. and Brémaud, P. (1994). *Elements of queueing theory: Palm-martingale calculus and stochastic recurrences*. Springer, Berlin.

Bartlett, M. S. (1955). *Stochastic processes, with special reference to methods and applications* (2nd ed. 1966, 3rd ed. 1978), Cambridge Univ. Press.

Bertoin, J. (1996). *Lévy processes*. Cambridge Tracts in Math. **121**, Cambridge Univ. Press.

Billingsley, P. (1979). *Probability and measure* (2nd ed. 1986, 3rd ed. 1 995). Wiley, New York.

Bingham, N. H. (2000). Measure into probability. From Lebesgue to Kolmogorov. Studies in the history of statistics and probability **XLVI**, *Biometrika* **87**, 145-156.

Bingham, N. H. and Kiesel, R. (1998). *Risk-neutral valuation. Pricing and hedging of financial derivatives* (2nd ed. 2004). Springer-Verlag, London.

Blumenthal, R. M. and Getoor, R. K. (1968). *Markov processes and potential theory*. Academic Press, New York.

Breiman, L. (1968). *Probability*. Addison-Wesley, Reading, MA.

Brémaud, P. (1981). *Point processes and queues. Martingale dynamics*. Springer, New York.

Brémaud, P. (1999). *Markov chains: Gibbs fields, Monte Carlo simulation, and queues*. Springer, New York.

Brockwell, P. J. and Davis, R. A. (1987). *Time series: Theory and methods*. Springer-Verlag, New York.

Chung, K.-L. (1960). *Markov chains with stationary transition probabilities* (2nd ed. 1967). Springer-Verlag, New York.

Chung, K.-L. (1968). *A course in probability theory* (2nd ed. 1973, 3rd ed. 2001). Academic Press, Reading, MA.

Cohn, H. (ed.) (1993). *Doeblin and modern probability.* Amer. Math. Soc., Providence RI.
Cramér, H. (1937). *Random variables and probability distributions.* Cambridge Tracts in Math. **36** (2nd ed. 1962, 3rd ed. 1970). Cambridge Univ. Press.
Cramér, H. (1976). Half a century with probability. Some personal recollections. *Ann. Probab.* **4**, 509-546.
Daley, D. J. and D. Vere-Jones (1988). *An introduction to the theory of point processes.* Springer, New York.
Davis, M. H. A. (1977). *Linear estimation and stochastic control.* Chapman and Hall, London.
Doob, J. L. (1949). Time series and harmonic analysis. *Proc. Berkeley Symp. Math. Statist. Probab.* 303-343. Univ. California Press, Berkeley CA.
Doob, J. L. (1953). *Stochastic processes.* Wiley, New York.// Doob, J. L. (1984). *Classical potential theory and its probabilistic counterpart.* Grundlehren math. Wiss. **262**, Springer-Verlag, New York.
Doob, J. L. (1990). Review of Masani (1990). *Bull. London Math. Soc.* **22**, 621-623.
Durrett, R. (1984). *Brownian motion and martingales in analysis.* Wadsworth, Belmont CA.
Dynkin, E. B. (1965). *Markov processes*, Volumes 1,2. Springer-Verlag, Heidelberg.
Ethier, S. N. and T. G. Kurtz (1986). *Markov processes. Characterization and convergence.* Wiley, New York.
Feller, W. (1950). *An introduction to probability theory and its applications*, Volume 1 (2nd ed. 1957, 3rd ed. 1968). Wiley, New York.
Feller, W. (1966). *An introduction to probability theory and its applications*, Volume 2 (2nd ed. 1971). Wiley, New York.
Gerber, H. U. (1986). *Lebensversicherungsmathematik.* Springer, Berlin.
Gihman, I. I. and Skorohod, A. V. (1969). *Introduction to the theory of random processes.* W. B. Saunders, Philadelphia PA.
Gihman, I.I. and Skorohod, A. V. (1974, 1975, 1979). *The theory of stochastic processes*, Volumes I, II, III. Springer-Verlag, New York.
Gnedenko, B. V. and Kolmogorov, A. N. (1954). *Limit theorems for sums of independent random variables.* Addison-Wesley, Reading MA.
Grimmett, G. R. and Stirzaker, D. (1982). *Probability and random processes* (2nd ed. 1992, 3rd ed. 2001), Oxford University Press.

Hall. P. G. and Heyde, C. C. (1980). *Martingale limit theory and its applications.* Academic Press, Reading, MA.
Harrison, J. M. and Pliska, S. R. (1981). Martingales and arbitrage in multiperiod securities markets. *Stochastic Processes and Applications* **11**, 215-260.
Halmos, P. R. (1950). *Measure theory.* New York, Van Nostrand.
Heyde, C. C. (1997). *Quasi-likelihood and its applications. A general approach to optimal parameter estimation.* Springer, New York.
Ibragimov, I. A. and Rozanov, Yu. A. (1978). *Gaussian random processes.* Springer-Verlag, New York.
Ikeda, N. and Watanabe, S. (1981). *Stochastic differential equations and diffusion processes.* North-Holland, Amsterdam.
Itô, K. and McKean, H. P. (1965). *Diffusion processes and their sample paths* (2nd ed. 1974). Springer-Verlag, New York.
Jacod, J. and A. N. Shiryaev (1987). *Limit theorems for stochastic processes.* Springer, Berlin.
Jagers, P. (1975). *Branching processes with biological applications.* Wiley, London.
Janson, S. (1997). *Gaussian Hilbert spaces*, Cambridge Tracts in Math. **129**. Cambridge Univ. Press.
Kaczmarz, S. and Steinhaus, H. (1935). *Theorie der Orthogonalreihen* (2nd ed. 1951, Chelsea, New York). Monogr Math., Warsaw-Lwow
Kakutani, S. (1944). Two-dimensional Brownian motion and harmonic functions. *Proc. Imp. Acad. Tokyo* **20**, 648-652.
Kahane, J.-P. (1985). *Some random series of functions*, 2nd ed. Cambridge Univ. Press (1st ed. 1968).
Karatzas, I. and S. E. Shreve (1988). *Brownian motion and stochastic calculus.* Springer, New York.
Karlin, S. and Taylor, H. (1975). *A first course in stochastic processes.* Academic Press, Reading MA.
Karlin, S. and Taylor, H. (1981). *A second course in stochastic processes.* Academic Press, Reading MA.
Kingman, J. F. C. and Taylor, S. J. (1966). *An introduction to measure and probability.* Cambridge Univ. Press.
Kolmogorov, A. N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Springer-Verlag, Berlin.
Krengel, U. (1985). *Ergodic theorems.* Walter de Gruyter, Berlin.
Lévy, P. (1925). *Calcul des probabilités.* Gauthier-Villars, Paris.

Lévy, P. (1937). *Théorie de l'addition des variables aléatoires* (2nd ed. 1954). Gauthier-Villars, Paris.
Lévy, P. (1948). *Processus stochastiques et mouvement brownien* (2nd ed. 1967). Gauthier-Villars, Paris.
Lindvall, T. (1991). W. Doeblin (1915-1940). *Ann. Probab.* **19**, 929-934.
Loève, M. (1973). Obituary: Paul Lévy, 1886-1971. *Ann. Probab.* **1**, 1-18.
Loève, M. (1955). *Probability theory* (2nd ed. 1960, 3rd ed. 1963). Van Nostrand, New York (4th ed., Volume I, Springer-Verlag 1977, Volume II, Springer-Verlag 1978, New York).
Lynch, J. D. (2000). The Galton-Watson process revisited: Some martingale relationships and applications. *J. Appl. Probab.* **37**, 322-328.
Lyons, T. J. and Qian, Z. (2002). *System control and rough paths.* Oxford Univ. Press.
Masani, P. R. (1990). *Norbert Wiener.* Birkhäuser, Basel.
Meyer, P.-A. (1966). *Probability and potentials.* Blaisdell, Waltham MA.
McKean, H. P. (1969). *Stochastic integrals.* Academic Press, New York.
Meyer, P.-A. (1976). Un cours sur les intégrales stochastiques. *Séminaire de Probabilités* **XI**, *Lecture Notes in Math.* **511**, 245-398, Springer-Verlag, Heidelberg.
Meyn, S. P. and Tweedie, R. L. (1993). *Markov chains and stochastic stability.* Springer-Verlag, New York.
Neveu, J. (1975). *Discrete-parameter martingales.* North-Holland, Amsterdam.
Nummelin, E. (1984). *General irreducible Markov chains and non-negative operators.* Cambridge Univ. Press.
Øksendal, B. (1985). *Stochastic differential equations. An introduction with applications* (2nd ed. 1989). Springer-Verlag, New York.
Protter, P. (1990). *Stochastic integration and differential equations: A new approach* (2nd ed. 1992). Springer-Verlag, New York.
Révész, P. (1968). *The laws of large numbers.* Academic Press, New York.
Revuz, D. (1975). *Markov chains* (2nd ed. 1984). North-Holland, Amsterdam.
Revuz, D. and M. Yor (1991). *Continuous martingales and Brownian motion* (2nd ed. 1994, 3rd ed. 1999). Springer, Berlin.
Rogers, L. C. G. (1989). Review of Sharpe (1988). *Bull. London Math. Soc.* **21**, 622-623.
Rogers, L. C. G. and Williams, D. (1994). *Diffusions, Markov processes and martingales. Volume 1: Foundations* (1st ed., D. Williams, 1979). Wiley,

Chichester.
Rogers, L. C. G. and Williams, D. (1987). *Diffusions, Markov processes and martingales. Volume 2: Itô calculus.* Wiley, Chichester.
Sato, K.-I. (1999). *Infinite divisibility and Lévy processes.* Cambridge Univ. Press.
Sharpe, M. J. (1988). *General theory of Markov processes.* Wiley, New York.
Spitzer, F. (1964). *Principles of random walk.* Van Nostrand, Princeton NJ.
Stone, M. H. (1932). *Linear transformations in Hilbert space and their applications to analysis.* AMS Colloquium Publications **15**. American Math. Soc, Providence RI.
Stroock, D. W. and Varadhan, S. R. S. (1979). *Multidimensional diffusions.* Springer-Verlag, New York.
Whittle, P. (1963). *Prediction and regulation by linear least-squares methods.* English Universities Press, London.
Wiener, N. (1949). *Extrapolation, interpolation and smoothing of stationary time series.* Wile, New York.
Zygmund, A. (1935). *Trigonometric series.* Monogr. Math., Warsaw-Lwow (2nd ed., 1959, Cambridge Univ. Press).

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