m3pm16soln7(14).tex

M3PM16/M4PM16 SOLUTIONS 7. 5.3.2015

Q1 *Prime divisor functions.* If $n = p_1^{r_1} \dots p_k^{r_k}$, $m = q_1^{s_1} \dots q_\ell^{s_\ell}$,

$$\Omega(mn) = r_1 + \dots + r_k + s_1 + \dots + s_\ell = \Omega(m)\Omega(n)$$

for all m, n, while $\quad \omega(mn) = k + \ell = \omega(m) + \omega(n) \quad$ if $(m, n) = 1$.
So $(-)^\Omega(mn) = (-)^{\Omega(m)}.(-)^{\Omega(n)}$ for all m, n, $(-)^{\omega(mn)} = (-)^{\omega(m)}.(-)^{\omega(n)}$ for m, n coprime. So $(-)^\Omega$ is completely multiplicative, $(-)^\omega$ is multiplicative.

Q2 *The Liouville function* λ. (i) For $n = p$ prime, $\Omega(p) = -1$, so $\lambda(p) = -1$. By Q1, λ is completely multiplicative. So by II.7 L11 (Squares and squarefree numbers: J p.67): take $a(n) := \lambda(n)/n^s$. Then as a is completely multiplicative as λ is,

$$\frac{\sum \lambda(n)/n^s}{\sum \lambda^2(n)/n^{2s}} = \prod_p \Big(1 + \frac{\lambda(p)}{p^s}\Big) = \prod_p \Big(1 - \frac{1}{p^s}\Big) = 1/\zeta(s),$$

by the Euler product (II.5 L9). As $\lambda^2 \equiv 1$, $\sum \lambda^2(n)/n^{2s} = \sum 1/n^{2s} = \zeta(2s)$:

$$\sum_{n=1}^\infty \lambda(s)/n^s = \zeta(2s)/\zeta(s).$$

(ii) By II.7 L9 again, $\quad \sum_{n=1}^\infty |\mu|(s)/n^s = \zeta(s)/\zeta(2s)$.
The product of these two Dirichlet series is 1. So in convolution language,

$$\lambda * |\mu| = \delta.$$

Q3. With $\nu(n) := \mu(d)$ if $n = d^2$ is a square, 0 otherwise:
Given n, extract from n the product of its prime factors raised to their highest *even* power. This is a square, m^2 say, and then $n = m^2 q$, with q a product of distinct primes (those occurring in n with *odd* multiplicity). So $|\mu|(q) = 1$. So $|\mu(n)| = \delta(m)$, since if $m = 1$, $n = q$, so $|\mu(n)| = |\mu(q)| = 1$, while if $m > 1$, n has a square factor, so both sides are 0. But $\mu * \mathbf{1} = \delta$, so

$$\begin{aligned} |\mu(n)| = \delta(m) = \sum_{d|m} \mu(d) &= \sum_{d^2|n} \mu(d) \qquad (d|m \text{ iff } d^2|n) \\ &= \sum_{d^2|n} \nu(d^2) \qquad (\text{definition of } \nu) \\ &= \sum_{d|n} \nu(d) \qquad (\nu(d) = 0 \text{ if } d \text{ is not a square}). \end{aligned}$$

This says that $|\mu| = \nu * \mathbf{1}$. Or: $|\mu| = \mu^2 = I_Q$ has Dirichlet series $\zeta(s)/\zeta(2s)$ (II.7 L11); $\mathbf{1}$ has Dirichlet series $\zeta(s)$; ν has Dirichlet series

$$\sum_n \nu(n)/n^s = \sum_d \mu(d)/d^{2s} = 1/\zeta(2s),$$

as $\nu(n) := \mu(d)$ if $n = d^2$, 0 otherwise.
Q4 (HW §18.6, Th. 333). (i) For integers $n \le y^2$, let $S(d)$ be the set of n with biggest square factor d^2. So $S(1)$ is the set of *square-free* $n \le y^2$. Then

$$|S(d)| = Q(y^2/d^2):$$

for $n \in S(d)$ iff $n = d^2 m \le y^2$ with m square-free, i.e. $m \le y^2/d^2$ is square-free, and there are $Q(y^2/d^2)$ such m, so this many $n = d^2 m$.

As $Q(x) = 0$ for $x < 1$, $Q(y^2/d^2) = 0$ for $d > y$, i.e. $S(d)$ is empty for $d > y$. So as the $S(d)$ form a partition of $\{n \le y^2\}$,

$$[y^2] = \sum_{d \le y} Q(y^2/d^2): \qquad [m] = \sum_{m \le \sqrt{x}} Q(x/m^2).$$

(ii) By Möbius inversion,

$$Q(y^2) = \sum_{d \le y} \mu(d)[y^2/d^2]: \qquad Q(x) = \sum_{m \le \sqrt{x}} \mu(m)[x/m^2].$$

(iii) Write $[.] = . - \{.\} = . + O(1)$:

$$Q(y^2) = \sum_{d \le y} \mu(d)(y^2/d^2 + O(1)) = \sum_{d=1}^{\infty} \mu(d)/d^2 + O(y^2 \sum_{d>y} 1/d^2) + O(y),$$

as $|\mu| \le 1$. For large y, $\sum_{d>y} 1/d^2 \sim \int_y^\infty dx/x^2 = 1/y$. So both error terms are $O(y)$, and we can combine them. As μ has Dirichlet series $1/\zeta$, $\sum_{d=1}^\infty 1/d^2 = 1/\zeta(2)$. But $\zeta(2) = \pi^2/6$ (Basel problem: Problems 4 Q3), so

$$Q(y^2) = \frac{6}{\pi^2} y^2 + O(y).$$

Replacing y by $\sqrt{x}$ gives the result. // NHB