m3pm16soln4.tex

M3PM16/M4PM16 SOLUTIONS 4. 14.2.2014

Q1 (HW §§5.5, 16.1,2, J, 68-9, A, §§2.3 - 2.5). (i) Using $|.|$ for cardinality, we partition the set $S := \{1, 2, \ldots, n\}$ as a disjoint union of the sets $A(d)$ containing those elements k of S whose gcd with n is d. So $\sum_1^n |A(d)| = n$. But $(k, n) = d$ iff k/d and n/d are coprime, and $0 < k \le n$ iff $0 < k/d \le n/d$. So if $q := k/d$, there is a one-one correspondence $k \leftrightarrow q = k/d$ between the elements of $A(d)$ and the integers q with $0 < q \le n/d$ with q and n/d coprime. The number of such q is $\phi(n/d)$ (definition of ϕ). So

$$\sum_{n|d} \phi(n/d) = n :$$

$$I = \phi * \mathbf{1} : \qquad \sum_{d|n} \phi(d) = n.$$

(ii) Since μ and $\mathbf{1}$ are convolution inverses, this gives

$$I * \mu = \phi * \mathbf{1} * \mu = \phi : \qquad \phi(n) = \sum_{d|n} \mu(d) I(n/d) = \sum_{d|n} \mu(d).n/d.$$

(iii) Since μ and I are multiplicative, this shows that $\phi = \mu * I$ is multiplicative. Taking Dirichlet series, as $\mu(n)$, $I(n) = n$ have Dirichlet series $1/\zeta(s) = \sum_1^\infty \mu(n)/n^s$, $\zeta(s-1) = \sum_1^\infty n/n^s = \sum_1^\infty 1/n^{s-1}$, this gives the Dirichlet series of ϕ as

$$\sum_1^\infty \phi(n)/n^s = \zeta(s-1)/\zeta(s).$$

(iv) Being multiplicative, ϕ is determined by its values on prime powers p^c, as prime powers of distinct primes are coprime. There are $p^c - 1$ positive integers $< p^c$, of which the multiples of p are $p, 2p, \ldots, p^c - p$ (so $p^{c-1} - 1$ of these), and the rest are coprime to p^c. So

$$\phi(p^c) = (p^c - 1) - (p^{c-1} - 1) = p^c(1 - \frac{1}{p}).$$

So if $n = \prod p^c$ is the prime-power factorisation of n (FTA), (ii) gives

$$\phi(n) = \prod \phi(p^c) = \prod p^c \prod (1 - \frac{1}{p}) = n \prod_{p|n} (1 - \frac{1}{p}). \qquad //$$

Q2 (HW Th. 260). (i) If $a \in A$ belongs to exactly m of the sets: if $m = 0$, a is counted in the RHS $S - S_1 + S_2...$ just once, in S itself. If $m > 0$, then a is counted:
$1 = \binom{m}{0}$ times in S; $\binom{m}{1}$ times in S_1, ..., $\binom{m}{r}$ times in S_r.
Altogether, a is counted

$$\binom{m}{0} - \binom{m}{1} + \binom{m}{2} \ldots = (1-1)^m = 0$$

times. So the RHS is the cardinality of the set of points in none of the A_i.
(ii) (HW Th. 261). The number of integers $\leq n$ and divisible by a is $[n/a]$. If a is coprime to b, the number of integers $\leq n$ and divisible by both a and b is $[n/ab]$, etc. So the number of integers $\leq n$ and not divisible by any of a coprime set of integers $a, b, \ldots$ is $[n] - \sum[n/a] + \sum[n/ab] \ldots$.
Taking $a, b, \ldots$ as the prime divisors of n,

$$\phi(n) = n - \sum \frac{n}{p} + \sum \frac{n}{pq} \ldots = n \prod_{p|n} \Big(1 - \frac{1}{p}\Big).$$

Q3 (see e.g. R. V. Churchill, *Fourier series and boundary value problems*, McGraw-Hill 1963, Ch. 4). Write a_n for the Fourier cosine coefficients of $|x|$ on $[-\pi, \pi]$ ($|.|$ is even, so we do not need sine terms). Then

$$\frac{1}{2}a_0 = \frac{1}{2}.\frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi}\int_0^{\pi} x dx = \frac{1}{\pi}[\frac{1}{2}x^2]_0^{\pi} = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos\ nx dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx = \frac{2}{n\pi}\int_0^{\pi} x d \sin nx$$

$$= \frac{2[x \sin nx]_0^{\pi}}{n\pi} - \frac{2}{n\pi}\int_0^{\pi} \sin nx dx = \frac{2}{n^2\pi}[\cos nx]_0^{\pi} = \frac{2(\cos n\pi - 1)}{n^2\pi}$$

$$= \frac{2((-1)^n - 1)}{n^2\pi} = -\frac{4}{\pi n^2}$$

if n is odd, 0 if n is even. So

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos(2m-1)x}{(2m-1)^2} : \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{odd} 1/n^2 : \quad \sum_{odd} = \pi^2/8.$$

But

$$\zeta(2) = \sum_1^{\infty} 1/n^2 = \sum_{odd} + \sum_{even} = \sum_{odd} + \frac{1}{4}\zeta(2) : \qquad \frac{3}{4}\zeta(2) = \frac{\pi^2}{8}, \qquad \zeta(2) = \frac{\pi^2}{6}.$$

NHB