m3pm16l8.tex

Lecture 8. 28.1.2015

§3. Holomorphy

Theorem. A Dirichlet series is holomorphic within its half-plane of convergence, with derivative given by termwise differentiation. If $F(s) := \sum_1^\infty a_n/n^s$ for $\sigma > \sigma_c$, then $F'(s) = -\sum_1^\infty \log n \ a_n/n^s$.

Proof. Choose $\alpha > \sigma_c$ and write $b_n := a_n/n^\alpha$. As above, b_n is bounded (by M, say). Write $F(s) = G(s-\alpha), G(s) := \sum_1^\infty b_n/n^s$. Write $G_N(s) := \sum_1^N b_n/n^s = \sum_1^N b_n e^{-s\log n}$. Then $G_N'(s) = -\sum_1^N \log n b_n/n^s$.

Take $\delta > 0, R > 0, K > 0, \Gamma$ the rectangle with sides $\sigma = \delta, \sigma = K, t = \pm R$, E its interior. By II.1, (**),

$$|G(s) - G_N(s)| \leq \frac{M}{N^\sigma}(\frac{|s|}{\sigma} + 1).$$

For $s \in E$,

$$\frac{|s|}{\sigma} \leq \frac{\sigma + |t|}{\sigma} = 1 + \frac{|t|}{\sigma} \leq 1 + \frac{R}{\sigma}.$$

So

$$|G(s) - G_N(s)| \leq \frac{M}{N^\delta}(2 + \frac{R}{\delta}) \to 0 \qquad (N \to \infty),$$

uniformly on $\Gamma \cup E$, which is compact. As each G_N is holomorphic by I.2, G is holomorphic. As each s with $\sigma > 0$ is in some E, G is holomorphic on $\sigma > 0$, so F is holomorphic on $\sigma > \alpha$. Then $G_N' \to G'$ by I.2, so as $D(n^{-s}) = D(e^{-s\log n}) = -\log n \ n^{-s}$, $F'(s) = -\sum_1^\infty \log n \ a_n/n^s$. Similarly for Dirichlet integrals: if $I_X(s) := \int_1^X f(x)dx/x^{1+s}$, then $I_X'(s) = -\int_1^X f(x)\log x dx/x^{1+s}$ by differentiating under the integral sign. //

Example.

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s}, \qquad \zeta'(s) = -\sum_1^\infty \log n/n^s \qquad (\sigma > 1).$$

Hence as in I.4 L8 (Integral Test), integrating by parts,

$$\int_1^\infty \frac{\log x}{x^\sigma}dx = \frac{1}{(\sigma-1)^2} \qquad (\sigma > 1); \qquad -\zeta'(\sigma) \leq \frac{1}{(\sigma-1)^2}.$$

§4. Convolutions

Absolutely convergent series may be rearranged. So if

$$F_a(s) := \sum_1^\infty a_n/n^s, \qquad F_b(s) := \sum_1^\infty b_n/n^s,$$

then in the half-plane where both converge absolutely

$$F_a(s)F_b(s) = (\sum_{i=1}^\infty \frac{a_i}{i^s})(\sum_{j=1}^\infty \frac{b_j}{j^s}) = \sum_{ij} \frac{a_i b_j}{i^r j^s} = \sum_{n=1}^\infty \frac{c_n}{n^s},$$

where

$$c_n := \sum_{ij=n} a_i b_j = \sum_{i|n} a(i)b(n/i).$$

The series $c = (c_n)$ so obtained is called the *Dirichlet convolution* of a and b:

$$c = a * b.$$

Write $\delta := (\delta_{1n})$ (the Kronecker delta: 1 if $n = 1$, 0 otherwise). Then $a * \delta = a$: δ acts as an identity.

Dirichlet convolutions have the properties:

$a * b = b * a$ – commutativity;

$a * (b + c) = a * b + a * c$ – distributivity;

$a * (b * c) = (a * b) * c$ – associativity.

Note also: $\mathbf{1}$, where $\mathbf{1}(n) := 1$ for all $n \geq 1$, so $\mathbf{1}$ has Dirichlet series

$$\zeta(s) := \sum_1^\infty 1/n^s; \qquad (\mathbf{1}, \zeta)$$

$d := (d_n)$, the *divisor function*, where $d_n := \sum_{d|n} 1$ is the number of divisors of n. Then

$$(\mathbf{1} * \mathbf{1})_n = \sum_{d|n} \mathbf{1}(d)\mathbf{1}(n/d) = \sum_{d|n} 1 = d(n) : \qquad \mathbf{1} * \mathbf{1} = d.$$

So taking Dirichlet series, one has the important Dirichlet series

$$\zeta(s)^2 = \sum_{n=1}^\infty d_n/n^s. \qquad (d, \zeta^2)$$