m3pm16soln8(14).tex

M3PM16/M4PM16 SOLUTIONS 8. 21.3.2014

Q1: *Sum-function of* ϕ (J 85-86).

As $\phi = \mu * I$ (Problems 4 Q1(v)), $\phi(n) = \sum_{md=n} \mu(d)m$. So

$$\begin{aligned}
\sum_{n \le x} \phi(n) &= \sum_{d \le x} \mu(d) \sum_{m \le x/d} m \\
&= \sum_{d \le x} \mu(d).\frac{1}{2}[x/d]([x/d]+1) \\
&= \sum_{d \le x} \mu(d).\frac{1}{2}(x/d + O(1))(x/d + O(1)) \\
&= \frac{1}{2}x^2 \sum_{x \le d} \mu(d)/d^2 + O(x \sum_{d \le x} 1/d).
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{d \le x} \mu(d)/d^2 &= \sum_1^\infty \mu(d)/d^2 + O(\sum_x^\infty 1/d^2) \\
&= \frac{6}{\pi^2} + O(1/x),
\end{aligned}$$

$$\sum_{d \le x} 1/d = \log x + \gamma + o(1)$$

(I.4 L3). So

$$\sum_{n \le x} \phi(n) = \frac{1}{2}x^2(\frac{6}{\pi^2} + O(1/x)) + O(x \log x) = \frac{3}{\pi^2}x^2 + O(x \log x). \qquad //$$

Q2: *$Q(x)$ and PNT.*

By Problems 7 Q3, with $\nu(n) := \mu(d)$ if $n = d^2$, 0 otherwise,

$$|\mu| = \nu * u.$$

In DHI (II.9 L14), take $a = u$, sum-function $A = [.]$, $b = \nu$, sum-function

$$B(x) := \sum_{n \le x} \nu(n) = \sum_{d^2 \le x} \mu(d) = \sum_{d \le \sqrt{x}} \mu(d) = M(\sqrt{x}).$$

So DHI gives, for $1 < y < x$,

$$\begin{aligned} Q(x) &= \sum_{n \le x} |\mu|(n) = \sum_{n \le x} (u * \nu)(x) \\ &= \sum_{j \le y} M(\sqrt{x/j}) + \sum_{k \le x/y} \nu(k)[x/k] - [y]M(\sqrt{x/y}) \end{aligned}$$

$$= \sum_{j \le y} M(\sqrt{x/j}) + \sum_{d \le \sqrt{x/y}} \mu(d)[x/d^2] - [y]M(\sqrt{x/y}). \qquad (i)$$

Writing $[.] = . + \{.\} = . + O(1)$, the second term in (i) is

$$\begin{aligned} \sum_{d \le \sqrt{x/y}} \mu(d)(x/d^2 + O(1)) &= x \sum_{d \le \sqrt{x/y}} \mu(d)/d^2 + O(\sqrt{x/y}) \\ &= x(\sum_1^\infty \mu(d)/d^2 - \sum_{d > \sqrt{x/y}} ...) + O(\sqrt{x/y}) \end{aligned}$$

$$= \frac{6}{\pi^2}x - x \int_{\sqrt{x/y}} dM(u)/u^2 + O(\sqrt{x/y}). \qquad (ii)$$

But as $M(x) = o(x)$ (given),

$$\int_z^\infty dM(u)/u^2 = -M(z)/z^2 + 2\int_z^\infty M(u)du/u^3 = o(1/z) + \int_z^\infty o(1/u^2).du = o(1/z).$$

So the second term in (ii) is $x.o(\sqrt{y/x}) = o(\sqrt{xy}) = o_y(\sqrt{x})$. Similarly, $\sum_{j \le y} M(\sqrt{x/j}) = o_y(\sqrt{x})$. Combining,

$$Q(x) = \frac{6}{\pi^2}x + o_y(x) + O(\sqrt{x/y}).$$

Take the first (x-) term on the right to the left, divide by $\sqrt{x}$, and let $x \to \infty$. The right becomes $o_y(1) + O(1/\sqrt{y})$. We can then let $y \to \infty$, to get

$$\limsup_{x \to \infty} |Q(x) - \frac{6}{\pi^2}x|/\sqrt{x} = 0 : \qquad Q(x) = \frac{6}{\pi^2}x + o(\sqrt{x}). \qquad // \qquad \text{NHB}$$