m3pm16l3.tex

Lecture 3. 17.1.2014

Abel's Summation Formula. If f has a continuous derivative on $[y,x]$,

$$\sum_{y<r\le x} a_r f_r = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

Proof. Let $m=[y], x=[n]$, with $[\cdot]$ the integer part. Then $\sum_{y<r\le n} a_r f_r = \sum_{m+1}^n a_r f_r$. As $A(x) := \sum_{r\le x} a_r$, $A(t) = A(r)$ for $r \le t < r+1$. So

$$\begin{aligned}
\sum_{m+1}^{n-1} A_r(f_r - f_{r+1}) &= -\sum_{m+1}^{n-1} A(r)\int_r^{r+1} f'(t)dt \\
&= -\sum_{m+1}^{n-1}\int_r^{r+1} A(t)f'(t)dt \qquad \text{as } A \text{ is constant on } (r,r+1) \\
&= -\int_{m+1}^n A(t)f'(t)dt.
\end{aligned}$$

Similarly, for $n \le t \le x$ $A(t) = A(n)$, so

$$A(x)f(x) - A(n)f(n) = A(n)[f(x) - f(n)] = \int_n^x A(t)f'(t)dt,$$

and for $m \le t \le y$ $A(t) = A(m)$, so

$$A(m)f(m+1) - A(y)f(y) = A(m)[f(m+1) - f(y)] = \int_y^{m+1} A(t)f'(t)dt.$$

Now substitute into $(*)$ in the proof of Abel's Lemma for $A_n f_n - A_m f_{m+1}$. //

Stieltjes integrals. If α is non-decreasing, and we replace $x_{i+1} - x_i$ in the Riemann integral everywhere by $\alpha(x_{i+1}) - \alpha(x_i)$, we obtain the *Riemann-Stieltjes (RS)* integral. The integration-by-parts formula

$$\int_a^b f dg = f(b)g(b) - f(a)g(a) - \int_a^b g df$$

holds for Stieltjes integrals. When α is a step-function $\alpha(x) = \sum_{n\le x} a_n$ and $f(x) = \int^x f'(u)du$ is absolutely continuous, we recover the result above.

§4. The Integral Test and Euler's Constant

The Integral Test: If $f > 0$ and is monotonic decreasing on $[1, \infty]$, then:
(i) $\int\limits_1^\infty f(x)dx$ and $\sum\limits_1^\infty f(n)$ converge or diverge together;
(ii) $\sum\limits_1^n f(r) - \int_1^n f(x)dx \to l \in [0, f(1)]$ as $n \to \infty$.

Proof. (i) As f is monotonic, it is integrable on each $[1, x]$. If $n - 1 \leq x \leq n$,

$$f(n-1) \geq f(x) \geq f(n): \qquad f(n-1) \geq \int_{n-1}^n f(x)dx \geq f(n).$$

Sum from 1 to $n - 1$:

$$\sum_1^{n-1} f(r) \geq \int_1^n f \geq \sum_2^n f(r): \quad \sum_1^n f(r) - f(n) \geq \int_1^n f \geq \sum_1^n f(r) - f(1). \tag{$*$}$$

If $\sum_1^\infty f(r) < \infty$, the LH inequality gives $\int\limits_1^\infty f(x)dx < \infty$.
If $\int_1^\infty f(x)dx < \infty$, the RH inequality gives $\sum_1^\infty f(r) < \infty$. For (ii),

$$f(1) \geq \phi(n) := \sum_1^n f(r) - \int_1^n f \geq f(n) \geq 0.$$

Then by $(*)$,

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^n f(x)dx \leq 0, \qquad 0 \leq \phi(n) \leq f(1),$$

So $\phi(n)$ is bounded and decreasing, so it is convergent: $\phi(n) \downarrow l \in [0, f(1)]$. //

Taking $f(x) \equiv 1/x$, the limit is defined as *Euler's constant*, γ. Then [J]:

Corollary (Euler's Constant).

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log n \to \gamma \qquad (n \to \infty).$$

$$0 < \sum_1^N \frac{1}{n} - \log N < 1; \qquad \sum_1^N \frac{1}{n} = \log N + \gamma + \frac{1}{2N} + O(\frac{1}{2N}).$$