m3pm16l29.tex
Lecture 29. 14.3.2014.
Now
$$-2A(R) + u \leq u \leq 2A(R) - u$$
(the LH inequality as $A(R) > 0$, the RH as $u \leq A(R)$). So
$$|u| \leq |2A(R) - u| : \quad u^2 \leq (2A(R) - -u)^2 : \quad |g|^2 \leq 1 : \quad |g| \leq 1.$$
So by Schwarz's Lemma with $M = 1$,
$$|g(z)| \leq r/R \qquad (|z| = r).$$
Now
$$g = \frac{f}{2A - f} : \quad 2Ag - gf = f : \quad f(1+g) = 2Ag : \quad f = \frac{2Ag}{1+g}.$$
Using $|g| \leq r/R$ in the numerator and $|1+g| \geq |1 - r/R|$ in the denominator,
$$|f(z)| \leq \frac{2A(R).r/R}{(1 - r/R)} = \frac{2A(R)r}{R - r},$$
proving the result in Case I: $f(0) = 0$.
II. If $f(0) \neq 0$: apply I to $f(z) - f(0)$:
$$|f(z) - f(0)| \leq \frac{2r}{R-r} \max_{|z|=R} Re\{f(z) - f(0)\} \leq \frac{2r}{R-r}(A(R) + |f(0)|) :$$
$$|f(z) - f(0)| \leq \frac{2r}{R-r}A(r) + |f(0)|\Big(1 + \frac{2r}{R-r}\Big) = \frac{2r}{R-r}A(r) + |f(0)|\Big(\frac{R+r}{R-r}\Big) :$$
$$M(R) \leq \frac{2r}{R-r}A(r) + |f(0)|\Big(\frac{R+r}{R-r}\Big). \qquad //$$
3. The zero-free region.

We give the classical zero-free region of Hadamard and de la Vallée Poussin. We follow Titchmarsh [T], Th. 3.8, Montgomery and Vaughan [MV] 6.1.

Theorem. For some absolute constant $c > 0$, $\zeta(s)$ has no zeros in the region
$$\sigma \geq 1 - \frac{c}{\log t} \qquad (t \geq t_0). \qquad (ZFR)$$

Proof. For $\sigma > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{\log p}{p^{ms}}, \qquad -Re\ \frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{\log p}{p^{m\sigma}} \cos(mt \log p).$$

So as in III.4, for $\sigma > 1$ and γ real (w.l.o.g. ≥ 2),

$$-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4Re\frac{\zeta'(\sigma+i\gamma)}{\zeta(\sigma+i\gamma)} - Re\frac{\zeta'(\sigma+2i\gamma)}{\zeta(\sigma+2i\gamma)}$$

$$= \sum_{p,m} \frac{\log p}{p^{m\sigma}} \{3 + 4\cos(m\gamma \log p) + \cos(2m\gamma \log p)\} \geq 0,$$

as $\{...\} \geq 0$ by III.4. As ζ has a simple pole at 1 of residue 1, so does $-\zeta'/\zeta$ (III.9 L19).

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{(\sigma-1)} + O(1).$$

By the partial fraction expansion for $-\zeta'/\zeta$ and Stirling's formula,

$$-\frac{\zeta'(s)}{\zeta(s)} = O(\log t) - \sum_{\rho} \Big(\frac{1}{s-\rho} + \frac{1}{\rho} \Big).$$

So ($s = \sigma + it$, $\rho = \beta + i\gamma$)

$$-Re\frac{\zeta'(s)}{\zeta(s)} = O(\log t) - \sum_{\rho} \Big(\frac{\sigma-\beta}{(\sigma-\beta)^2 + (t-\gamma)^2} + \frac{\beta}{\beta^2+\gamma^2} \Big).$$

Each term in the last sum is positive (as $\frac{1}{2} \leq \beta < 1$, $\sigma > 1$). So

$$-Re\frac{\zeta'(s)}{\zeta(s)} < O(\log t) : \qquad -Re\frac{\zeta'(\sigma+2i\gamma)}{\zeta(\sigma+2i\gamma)} < O(\log \gamma).$$

Also, taking $s = \sigma + i\gamma$ with $\rho = \beta + i\gamma$ gives

$$-Re\frac{\zeta'(\sigma+i\gamma)}{\zeta(\sigma+i\gamma)} < O(\log \gamma) - \frac{1}{\sigma-\beta},$$

discarding every term (as above) except $1/(s-\rho)$. Combining,

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + O(\log \gamma) \geq 0 \qquad (\gamma \to \infty).$$