m3pm16l28.tex
Lecture 28. 14.3.2014.

We can also estimate Γ in vertical strips. For this, only the leading term $z^{z-\frac{1}{2}} = \exp\{(z-\frac{1}{2})\log z\}$ in Stirling's formula matters, and only large t matters. One obtains:

$$|\Gamma(\sigma+it)| <<_{\alpha,\beta} |t|^{\beta-\frac{1}{2}} e^{-\frac{1}{2}\pi t} \qquad (\alpha \leq \sigma \leq \beta,\ t > 1):$$

$|(z-\frac{1}{2})\log z| = (\sigma - \frac{1}{2})\log r - \theta t$; as $t \to \infty$, $r \sim t$, $\theta \uparrow \frac{1}{2}\pi$, so this is $<< \log(t^{\beta-\frac{1}{2}}.e^{-\frac{1}{2}\pi t})$.

Entire functions of order 1.

Hadamard, in the course of his proof of PNT using Complex Analysis in 1896, developed a theory of factorization of entire functions. This is standard Complex Analysis (see e.g. Titchmarsh [T2], 8.24 or Ahlfors [Ahl], 5.3.2) rather than Number Theory, so we shall quote what we need. The *order* of an entire function f is the least a for which

$$|f(z)| = O_\delta(\exp\{|z|^{a+\delta}\}) \qquad (|z| \to \infty).$$

We shall only need the case of *order 1*, and that only for Γ and ζ. Hadamard's factorization theorem for entire functions f of order 1 states that
(i) f can be written as

$$f(z) = z^r e^{Az+B} \prod_{\rho \neq 0} \{(1 - z/\rho)e^{z/\rho}\},$$

where r is the order of the zero at 0 (if any), A, B are constants, and ρ runs through the other zeros (if any);
(ii)

$$\sum_{\rho\neq 0} |\rho|^{-1-\delta} < \infty \qquad \forall\ \delta > 0.$$

Taking $\delta = 1$ in (ii) gives $\sum |\rho|^{-2}$ converges, whence the product in (i) converges. The proof involves Jensen's formula from Complex Analysis.

We have already met two instances of this, the product for sin (Problems 6 Q3) and Weierstrass's product definition of Γ (I.7). The product for ζ gives an alternative route to PNT with remainder (see 2013, III.9 L26: in 2013 we used distribution of zeros of ζ and the Riemann-von Mangoldt formula; this year we use Perron's formula and the Borel-Carathéodory theorem).

We quote ([AL], or [T2], §5.5):

Theorem (Maximum Modulus Principle). If f is holomorphic inside and on a contour Γ, and $|f| \leq M$ *on* Γ, then $|f| < M$ *inside* Γ – unless f is constant, $\equiv M$.

Theorem (Schwarz's Lemma). If f is holomorphic in $|z| \leq R$, $|f(z)| \leq M$ on $|z| = R$ and $f(0) = 0$, then

$$|f(re^{i\theta})| \leq Mr/R \qquad (0 \leq r \leq R).$$

The next result uses a *one-sided* upper bound on the *real* part to get an *O- bound* on the (maximum) *modulus*. This will be crucially useful applied to $-\zeta'/\zeta$ (IV.4) ((**) L31, and L32). It is due to Borel in 1897, Carathéodory (according to Landau in 1908).

Theorem (Borel-Carathéodory Inequality) ([T2] 5.5, MV 6.1). If f is holomorphic in $|z| \leq R$,

$$M(r) := \sup\{|f(z)| : |z| \leq r\},$$

$$A(r) := \sup\{Re\ f(z) : |z| \leq r\}$$

– then

$$M(r) \leq \frac{2r}{R-r}A(R) + \frac{R+r}{R-r}|f(0)| \qquad (0 < r < R).$$

Proof. The result holds if f is constant, so suppose f is not constant.
I. If $f(0) = 0$. Then $A(R) > A(0) = 0$. Write

$$g(z) := \frac{f(z)}{2A(R) - f(z)}.$$

If $f = u + iv$, the real part of the denominator is $2A(R) - u$, and as $A(R) > 0$ (above) and $A := \sup\ u$, this is non-zero (> 0). So the denominator is non-zero, so g is holomorphic. Also $g(0) = 0$ as $f(0) = 0$. As

$$g = \frac{u+iv}{2A-u-iv}, \qquad \bar{g} = \frac{u-iv}{2A-u+iv},$$

$$|g|^2 = g\bar{g} = \frac{u^2+v^2}{(2A-u)^2+v^2}.$$