m3pm16l23.tex

Lecture 23. 7.3.2014.

Proof of Newman's theorem (concluded): Slow decrease.

By slow decrease of ρ,

$$\int_T^{T+\delta} \{\rho(t) - \rho(T)\} dt \geq - \int_T^{T+\delta} \epsilon(t,T) dt,$$

where $\epsilon(t,T) \to 0$ ($t, T \to \infty$, $0 \leq t - T \to 0$). So

$$\delta\rho(T) \leq \int_T^{T+\delta} \rho(t) dt + \int_T^{T+\delta} \epsilon(t,T) dt.$$

The first term on RHS $\to 0$ (above), and the second is $\leq \epsilon\delta$ for T large. So

$$\limsup_{T\to\infty} \delta\rho(T) \leq \epsilon\delta : \qquad \limsup_{T\to\infty} \rho(T) \leq \epsilon : \qquad \limsup_{T\to\infty} \rho(T) \leq 0,$$

as $\epsilon > 0$ is arbitrary. Arguing similarly from $\int_{T-\delta}^{T} \ldots$ gives $\liminf \rho(T) \geq 0$. Combining,

$$\rho(T) \to 0 \qquad (T \to \infty). \qquad //$$

Note. The last part, using slow decrease above, is an *elementary Tauberian* argument. It rests on the assumption (iii) in Newman's Theorem 1, that $s_n = O(n)$. The result is true without this assumption, but is then harder to prove and this is all we need.

Newman's theorem is a variant on (or alternative to) the Wiener-Ikehara theorem (which we used in 2013, again following Korevaar, but then his book of 2004 rather than his papers of 2004 and 2006). By either of these routes, or any other of the shorter routes to PNT, one needs *some* Tauberian argument somewhere. For background on Tauberian theory, see the Handout on the website.

6. Proof of PNT.

In Th. 1 of §5, take

$$a_n := \Lambda(n) :$$

then

$$s_n = \psi(n).$$

By Chebyshev's Upper Estimate (III.2, L17),

$$\psi(n) = O(n).$$

So
$$\rho(.) = O(1).$$
As in the proof of Th. 1, this and s_n increasing (as $\Lambda(n) \geq 0$) give ρ slowly decreasing. The Dirichlet series $f(s)$ of a_n is $-\zeta'(s)/\zeta(s)$ (II.7, L12). As ζ has an analytic continuation to $\mathbb{C}$ and is analytic except for a simple pole of residue 1 (III.3 L19), and ζ is non-vanishing on the 1-line (III.4 L20),
$$-\zeta'(s)/\zeta(s) - \frac{1}{s-1}$$
is entire. This gives the convergence needed in Th. 2(ii) on each $[-iR, iR]$, however large R is (Th. 2(iii)). So Th. 2 applies, and so also Th. 1 with $A = 1$, giving
$$\psi(t)/t \to 1 \qquad (t \to \infty).$$
This is PNT (Equivalence Theorem, III.2 L17). //

7. Functional equation for the Riemann zeta function

Poisson Summation Formula.

We quote ([AL], or [Kat] VI.1.15) the *Poisson summation formula*:
$$2\pi\lambda \sum_{n=-\infty}^{\infty} f(2\pi\lambda n) = \sum_{n=-\infty}^{\infty} \hat{f}(n/\lambda). \qquad (PSF)$$

Theta function.

If we use this in (PSF), we find after a change of variables that
$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi/x}. \qquad (\theta)$$
This is one of Jacobi's identities for the *Jacobi theta function* (transformation under the modular group); see e.g. [WW] 21.51, or Apostol [A2], p.91, 141. It can be re-written as follows: if
$$\Psi(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x},$$
then
$$2\Psi(x) + 1 = \frac{1}{\sqrt{x}}(2\Psi(1/x) + 1). \qquad (\theta)$$