m3pm16l20.tex
Lecture 20. 28.2.2014

4. Non-vanishing on the 1-line: $\zeta(1+it) \neq 0$.

Lemma. $3 + 4\cos\theta + \cos 2\theta \geq 0$.

Proof. $3 + 4\cos\theta + \cos 2\theta = 2 + 4\cos\theta + 2\cos^2\theta = 2(1+\cos\theta)^2$. //

Prop. If all $a_n \geq 0$ and the Dirichlet series $f(s) := \sum_1^\infty a_n/n^s$ converges for $Re\ s = \sigma > \sigma_0$, then

$$3f(\sigma) + 4Ref(\sigma+it) + Ref(\sigma+2it) \geq 0 \qquad (\sigma > \sigma_0).$$

Proof.

$$3f(\sigma) + 4Ref(\sigma+it) + Ref(\sigma+2it) = \sum_1^\infty \frac{a_n}{n^\sigma}(3 + 4n^{-it} + n^{-2it}).$$

If $\theta_n := t\log n$, $Re(3 + 4n^{-it} + n^{-2it} = 3 + 4\cos\theta_n + \cos 2\theta_n \geq 0$. //

Corollary. For $\sigma > 1$ and all t,

$$H(\sigma) := \zeta(\sigma)^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+2it)| \geq 1.$$

Proof. By II.6, $\log\zeta(s)$ has a Dirichlet series with non-negative coefficients, $\log\zeta(s) = f(s) = \sum_1^\infty a_n/n^s$ for $a_n \geq 0$. By the Proposition, $3f(\sigma) + 4Ref(\sigma+it) + Ref(\sigma+2it) \geq 0$. So ($\log z = \log(re^{i\theta}) = \log r + i\theta$, so $Re\log z = \log r = \log|z|$)

$$3\log\zeta(\sigma) + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \geq 0.$$

Exponentiating gives the result. //

Theorem. $\zeta(1+it) \neq 0$ for $t \neq 0$.

Proof (by contradiction). If not, $\zeta(1+it) = 0$ for some $t \neq 0$. Then differentiating from first principles,

$$\frac{\zeta(\sigma+it) - \zeta(1+it)}{(\sigma+it)-(1+it)} = \frac{\zeta(\sigma+it)}{\sigma-1} \to \zeta'(1+it) \qquad (\sigma \downarrow 1),$$

as ζ is holomorphic at $1+it$. In the Corollary,

$$H(\sigma) = [(\sigma-1)\zeta(\sigma)]^3 \left(\frac{|\zeta(\sigma+it)|}{\sigma-1}\right)^4 [(\sigma-1)|\zeta(\sigma+2it)|]\,.$$

Now $(\sigma-1)\zeta(\theta) \to 1$ $(\sigma \downarrow 1)$ (ζ has a simple pole of residue 1 at 1). So $[...]^3 \to 1$; $(...)^4 \to (\zeta'(1+it))^4$ by above; $|\zeta(\sigma+2it)| \to \zeta(1+2it)$. Combining, $H(\sigma) \to 0$ as $\sigma \to 1$, contradicting the Corollary above. //

Note. 1. The critical term in the proof above is the factor $\sigma - 1$ in the last [...] (available because of the "3, 4, 1" coefficients in the Lemma (see below).
2. $\zeta(1+it) \neq 0$ is essentially equivalent to the PNT, below.

Recall: from the Euler product, $\zeta \neq 0$ to the *right* of the 1-line; by the Theorem, $\zeta \neq 0$ *on* the 1-line. We now extend the zero-free region of ζ to the *left* of the 1-line and into the *critical strip* of $0 \leq \sigma \leq 1$. It suffices to consider $t > 0$, as $|\zeta(\sigma - it)| = |\zeta(\sigma+it)|$ (since $n^{-s} = e^{-it\log n}/n^{\sigma}$).

Theorem. For $0 < a < b$, $\exists \delta > 0$ such that $\zeta(\sigma+it) \neq 0$ in $1-\delta \leq \sigma \leq 1$, $a \leq t \leq b$ (a rectangle *inside* the critical strip).

Proof. If not, for each n there exists some $s_n = \sigma_n + it_n$ with

$$1 - 1/n \leq \sigma_n \leq 1, \ a \leq t_n \leq b, \ \zeta(s_n) = 0.$$

As t_n is an infinite sequence in $[a,b]$, which is compact, it has a convergent subsequence t_{n_k} (Bolzano-Weierstrass Th.), going to t_0, say. Then $\sigma_n \to 1$, so $s_{n_k} \to 1 + it_0$. So $\zeta(s_{n_k}) \to \zeta(1+it_0)$ by the continuity of ζ, and this is non-zero by the Theorem above. But each $\zeta(s_n) = 0$, so $\zeta(s_{n_k}) = 0 \to 0$, a contradiction. //

Note. 1. The Lemma and proof above are due to Hadamard in his original proof of PNT in 1896. It is clear and efficient, but seems unmotivated (or like a 'trick'). For an approach which both seems more natural and is more general (non-vanishing of Dirichlet L-series, rather than just the zeta function), see Newman [N], VI: *A "natural" proof of the non-vanishing of L-series.*
2. $\zeta(1+it) \neq 0$ (which we use via $-\zeta'/\zeta$ holomorphic on the 1-line) is exactly what is needed to apply the most important Tauberian theorem, Wiener's Tauberian theorem; see 2013, III.10.3 and Handout on Tauberian theorems.