m3pm16l18.tex

Lecture 18. 21.2.2014

LEMMA 1.

$$\sum_{j \le x} \Lambda(j) E(x/j) = \sum_{k \le x} \log k - 2 \sum_{k \le x/2} \log k \quad (x \ge 2).$$

Proof. By the Lemma of II.3 (sum of a convolution),

$$\begin{aligned} S := \sum_{j \le x} \Lambda(j) E(x/j) &= \sum_{j \le x} [\Lambda * (u * \nu)](j) \qquad \text{(Lemma: } E \text{ sum-function of } u * \nu) \\ &= \sum_{j \le x} (\ell * \nu)(j) \qquad (\Lambda * \nu = \ell) \\ &= \sum_{j \le x} \nu(j) \sum_{k \le x/j} \log k \qquad (\ell = \log\text{; Lemma again}) \\ &= \sum_{k \le x} \log k - 2 \sum_{k \le x/2} \log k \quad (x \ge 2). \quad // \end{aligned}$$

LEMMA 2.

$$\psi(2n) \ge \log \binom{2n}{n}.$$

Proof. Take $x = 2n$ in Lemma 1. As each $E(.) \le 1$,

$$S \le \sum_{j \le 2n} \Lambda(j) = \psi(2n).$$

But

$$\sum_{k \le x} \log k - 2 \sum_{k \le x/2} \log k = \sum_{k=n+1}^{2n} \log k - \sum_{k=1}^{n} \log k = \log \Big(\frac{(n+1)(n+2)\dots(2n)}{1.2.\dots n} \Big) = \log \binom{2n}{n}. \quad //$$

Th. 3 (Chebyshev's Lower Estimates). For $\epsilon > 0$ and x large,
(i) $\psi(x) \ge (\log 2 - \epsilon) x$;
(ii) $\theta(x) \ge (\log 2 - \epsilon) x$;
(iii) $\pi(x) \ge (\log 2 - \epsilon) li(x)$.

Proof. (i) Let $N := \binom{2n}{n}$ as above. This is the largest of the $2n+1$ terms in the binomial expansion of $(1+1)^{2n} = 2^{2n}$ (by Pascal's triangle), so $2^{2n} \le (2n+1)N$. So by the Lemma above,

$$\psi(2n) \ge \log N \ge 2n \log 2 - \log(2n+1).$$

Given x, take n with $2n \leq x < 2n+2$. Then (i) follows as

$$\psi(x) \geq \psi(2n) \geq (x-2)\log 2 - \log(x+1).$$

(ii) This follows from (i) as $(\psi(x) - \theta(x))/x \to 0$ (Cor. above).
(iii) This follows from (ii) by the first Theorem of this section. //

Cor. 5. $\pi(x) \geq (\log 2 - \epsilon)x/\log x$.

Proof. $\psi(x) \leq \pi(x)\log x$ (first Prop. of this section and (i). //

THEOREM (Chebyshev, 1849-51) (Mastery Question, 2013).

$$\liminf \pi(x)/li(x) \leq 1 \leq \limsup \pi(x)/li(x).$$

In particular, if the limit exists, it is 1 (as in PNT).

Proof. For all $\epsilon > 0$ there exists x_0 such that for $x \geq x_0$

$$\ell - \epsilon \leq \frac{\pi(x)}{x/\log x}, \qquad \frac{\pi(x)}{x/\log x} \leq L + \epsilon.$$

For the lower bound, integration by parts gives, as $0 < \pi(u) \leq u$,

$$\sum_{p \leq x} \frac{1}{p} \geq \sum_{x_0 < p \leq x} \frac{1}{p} = \int_{x_0}^{x} \frac{d\psi(u)}{u} = \frac{\pi(x)}{x} - \frac{\pi(x_0)}{x_0} + \int_{x_0}^{x} \frac{\pi(t)}{t^2} dt,$$

$$\geq -1 + \int_{x_0}^{x} \frac{\pi(t)}{t^2} dt \geq -1 + (\ell - \epsilon)\int_{x_0}^{x} \frac{dt}{t \log t} \geq (\ell - \epsilon)\log\log x + O_\epsilon(1)$$

($\int^x dt/(t\log t) = \int^x d\log t/\log t = \log\log x$). But by Mertens' Second Th.,

$$\sum_{p \leq x} 1/p = \log\log x + c_1 + O(1/\log x).$$

Combining,

$$\log\log x + c_1 + O(1/\log x) \geq (\ell - \epsilon)\log\log x + O_\epsilon(1): \qquad 1 \geq \ell - \epsilon.$$

This holds for all $\epsilon > 0$. So $\ell \leq 1$. The upper bound is similar but simpler. //

In 1851, Chebyshev also proved *Bertrand's postulate* of 1845: for any $n \geq 2$ there is a prime p between n and $2n$; see 2013 Problems and Solutions 8 for Erdös' elementary proof of 1932.